The Relative Power of Synchronization Operations Queues and Stacks

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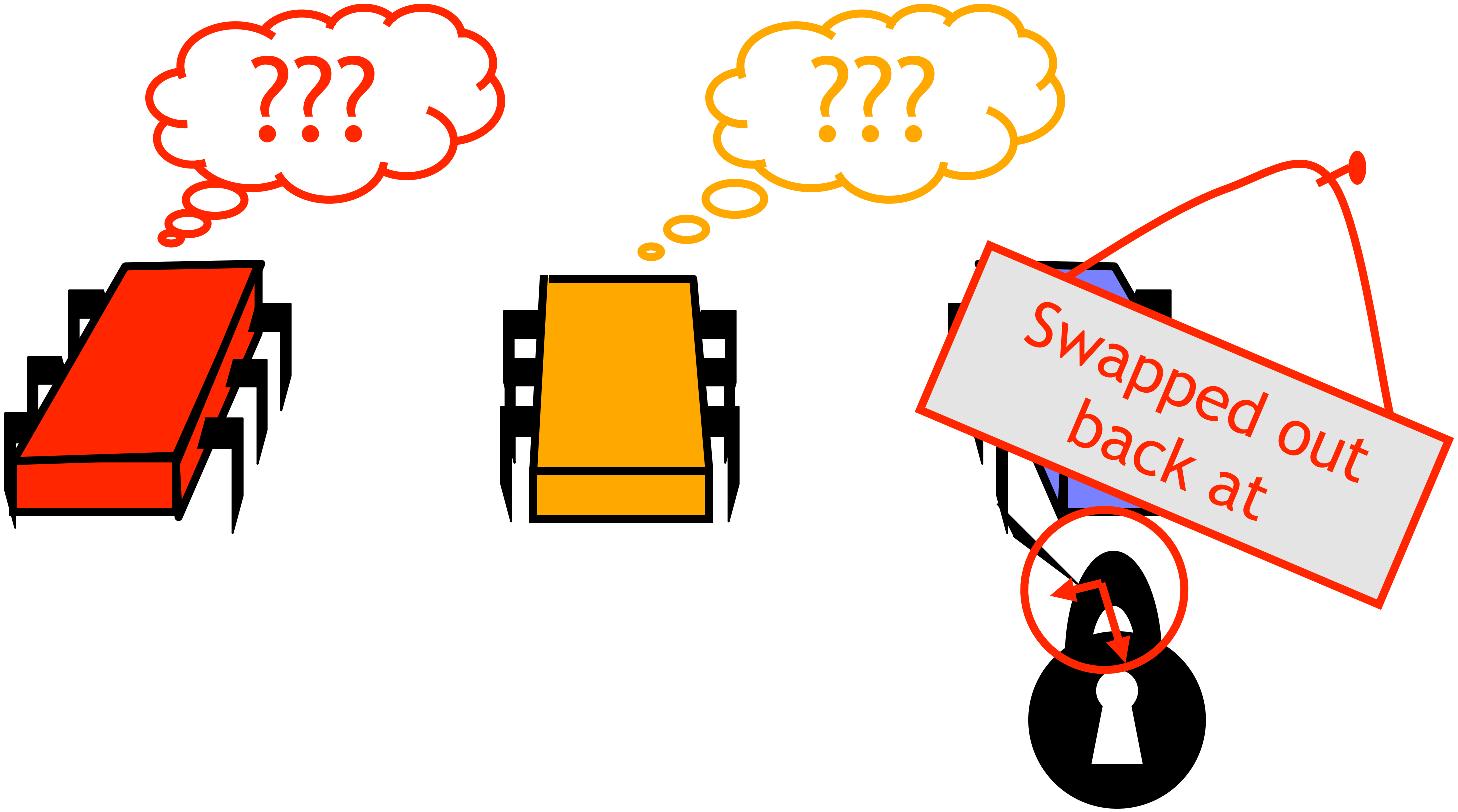
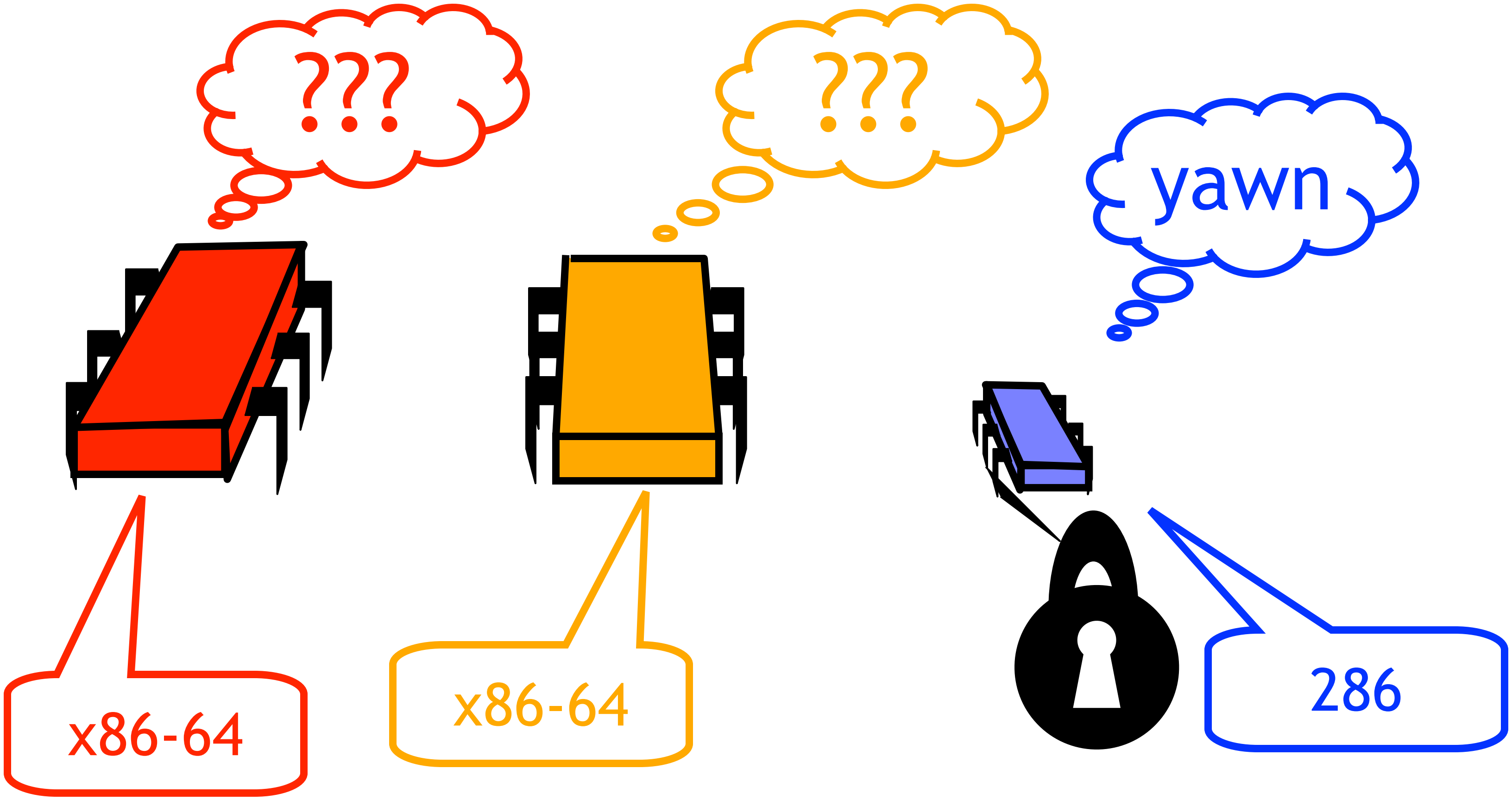
Based on slides by Maurice Herlihy and Nir Shavit

Why is Mutual Exclusion so wrong?

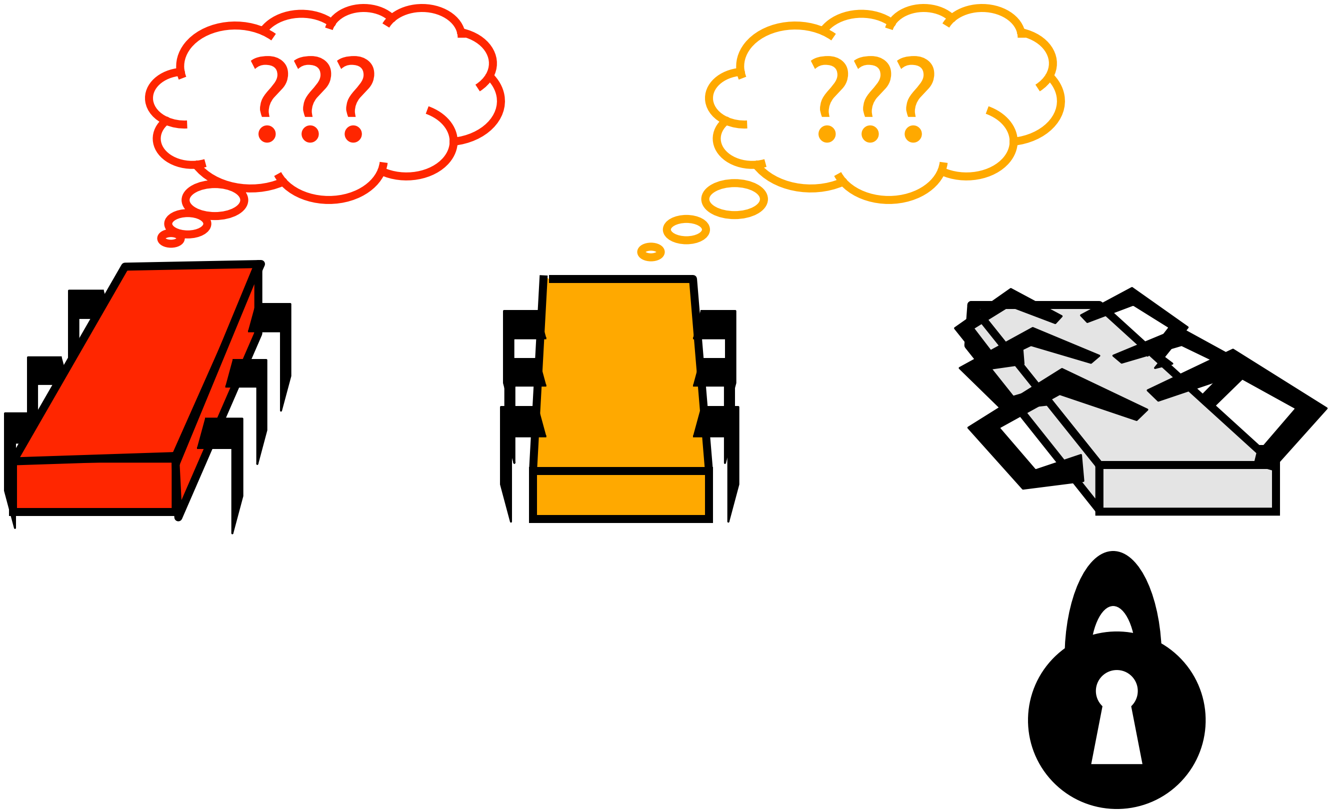
Asynchronous Interrupts



Heterogeneous Processors



Fault-tolerance



Wait-Free Implementations

Definition: An object implementation is **wait-free** if every thread completes a method in a finite number of steps

No mutual exclusion

- Thread could halt in critical section
- Build mutual exclusion from registers

Lock-Free Implementations

Definition: An object implementation is **lock-free** if in an infinite execution infinitely often some method call finishes (obviously, in a finite number of steps)

No difference between lock-free and wait-free for finite executions

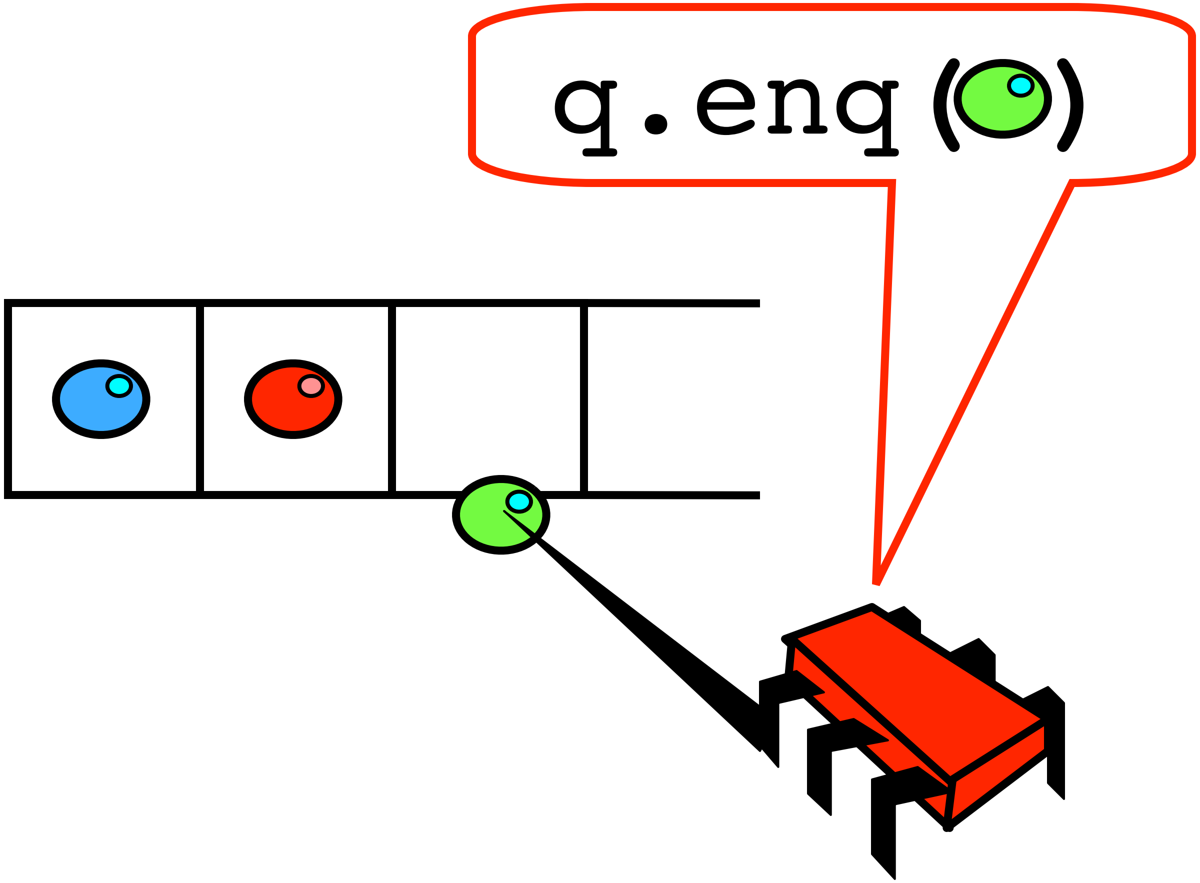
Basic Questions

- Wait-Free synchronization might be a good idea in principle
- But how do you do it
  - Systematically?
  - Correctly?
  - Efficiently?

Today: Focus on Wait-free

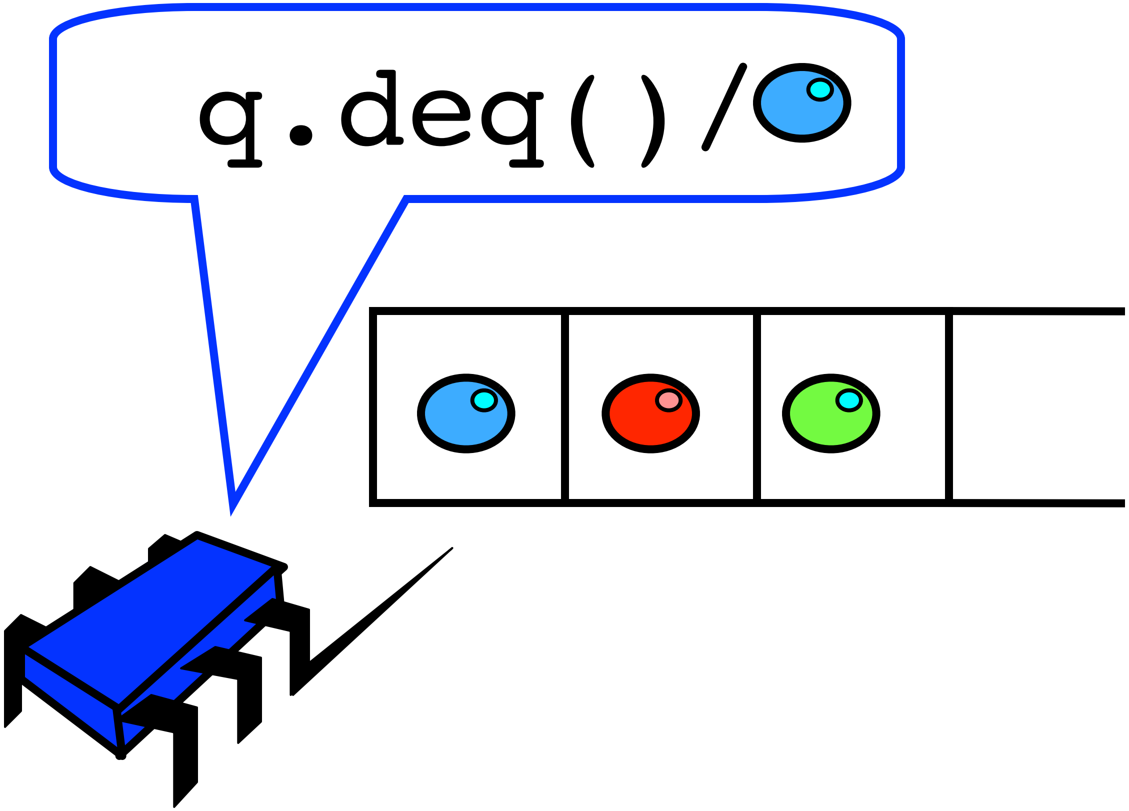
- The rest of today’s discussion will focus on wait-free implementations
- But the results we present apply almost verbatim to lock-free ones

FIFO Queue: Enqueue Method



`q.enq(●)`

FIFO Queue: Dequeue Method



`q.deq()/`●

Two-Thread Wait-Free Queue

```
public class LockFreeQueue {
 int head = 0, tail = 0;
 Item[QSIZE] items;
 public void enq(Item x) throws… {
  if (tail-head == QSIZE) { throw…};
  items[tail % QSIZE] = x; tail++;
  }
 public Item deq() {
  if (tail-head == 0) { throw…}
  Item item = items[head % QSIZE];
  head++; return item;
}}
```

Two-Thread Wait-Free Queue

```
public class LockFreeQueue {
 int head = 0, tail = 0;
 Item[QSIZE] items;
 public void enq(Item x) throws… {
  if (tail-head == QSIZE) { throw…};
  items[tail % QSIZE] = x; tail++;
  }
 public Item deq() {
  if (tail-head == 0) { throw… }
  Item item = items[head % QSIZE];
  head++; return item;
}}
```

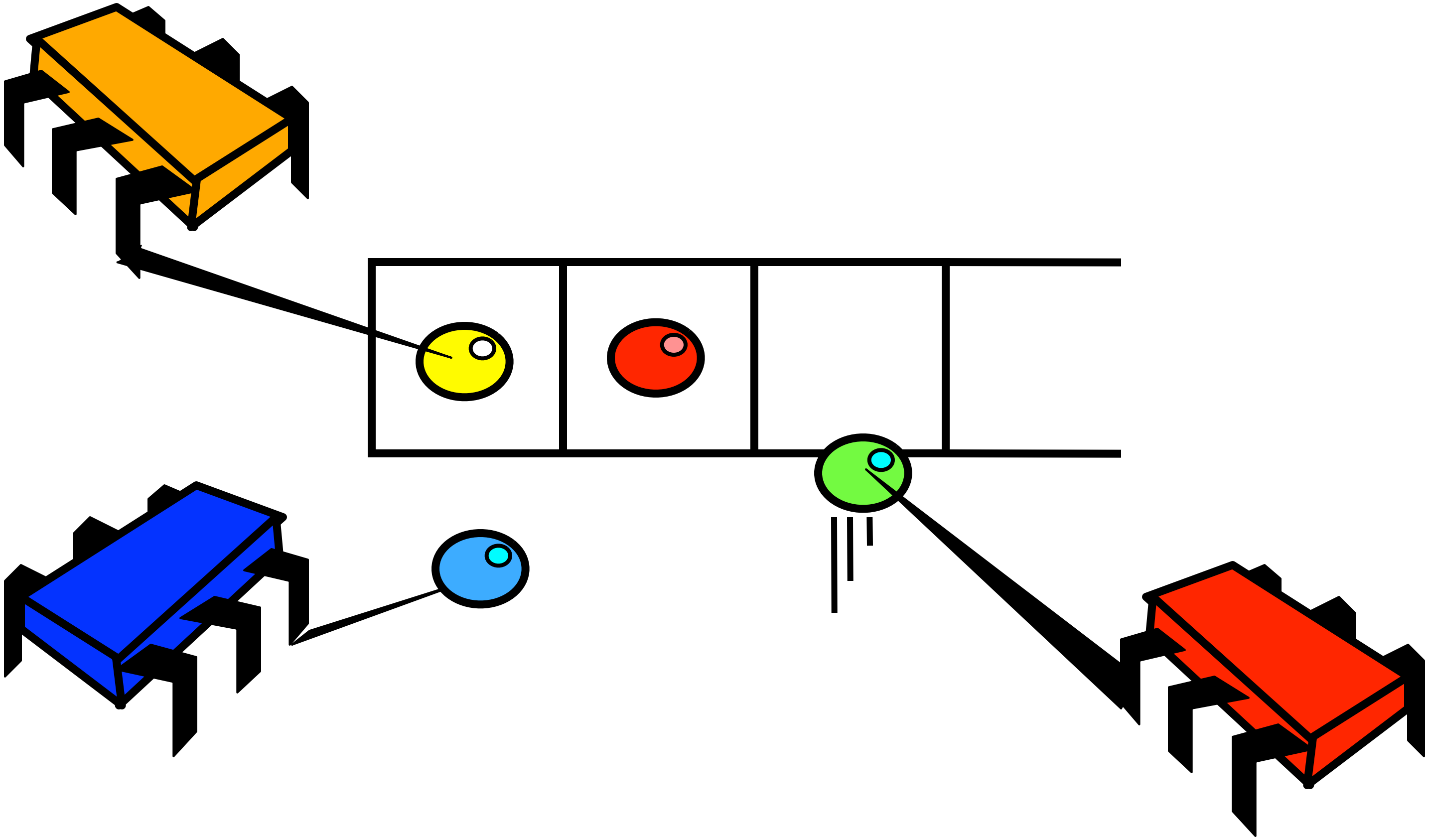
Put object in queue

Two-Thread Wait-Free Queue

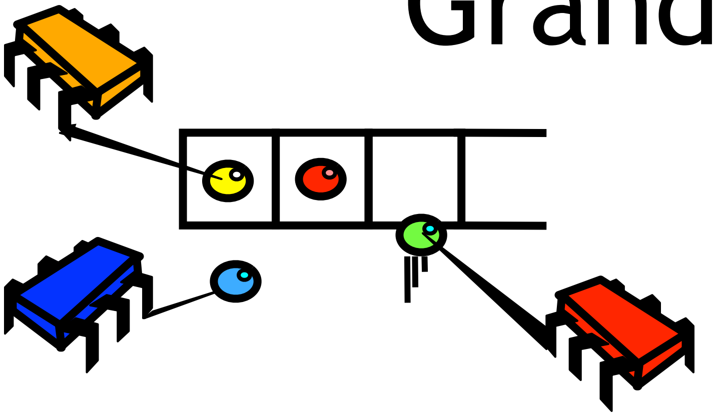
```
public class LockFreeQueue {
 int head = 0, tail = 0;
 Item[QSIZE] items;
 public void enq(Item x) throws… {
  if (tail-head == QSIZE) { throw…};
  items[tail % QSIZE] = x; tail++;
  }
 public Item deq() {
  if (tail-head == 0) { throw…}
  Item item = items[head % QSIZE];
  head++; return item;
}}
```

Increment tail counter

What About Multiple Dequeuers?



Grand Challenge



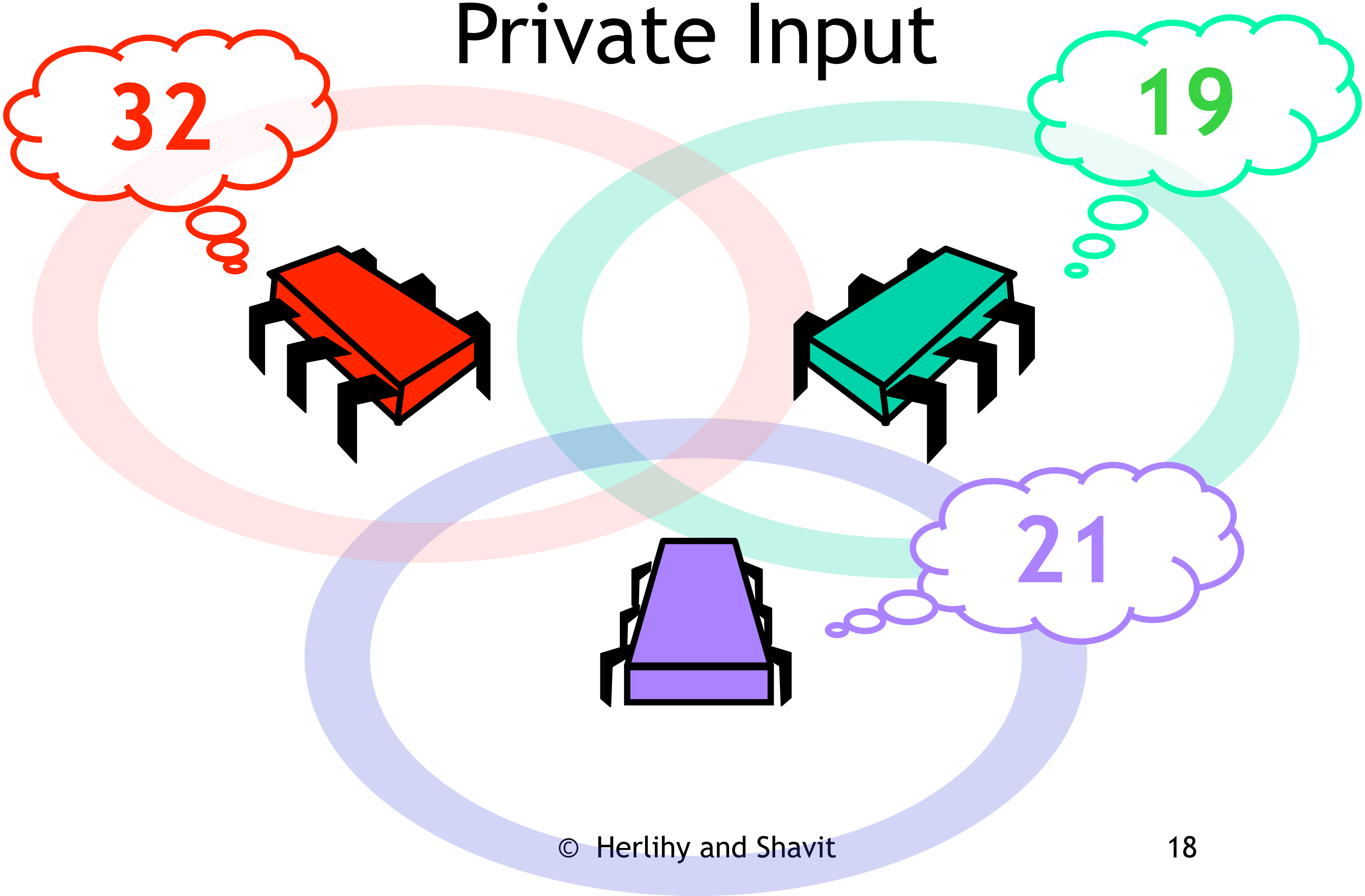
- Implement a FIFO queue
  - Wait-free
  - Linearizable
  - From atomic read-write registers
  - Multiple dequeuers

Only new aspect

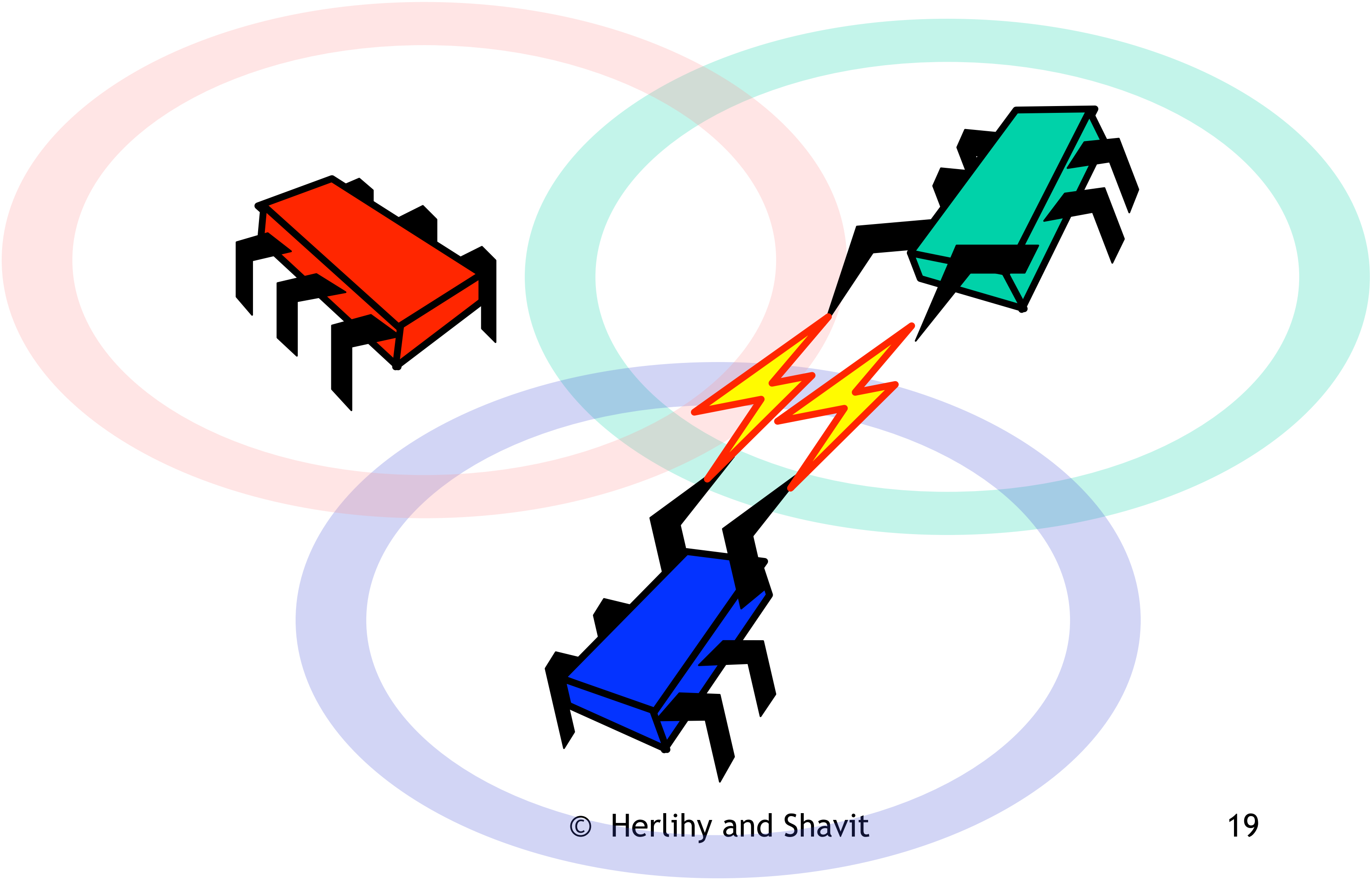
Consensus

- While you are thinking about the grand challenge…
- We will give you another puzzle
  - Consensus
  - Will be important …

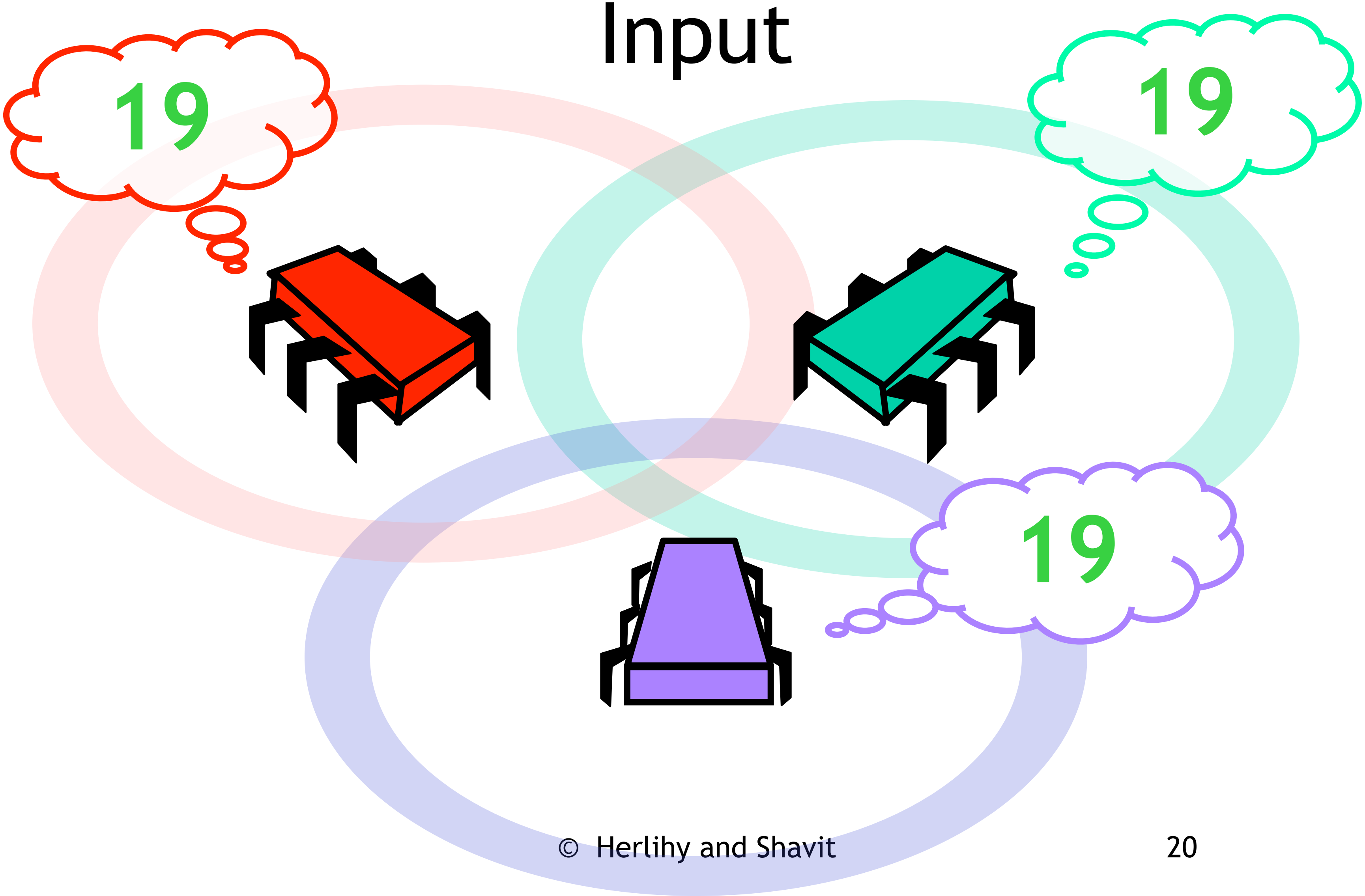
Consensus: Each Thread has a Private Input



They Communicate



They Agree on One Thread's Input



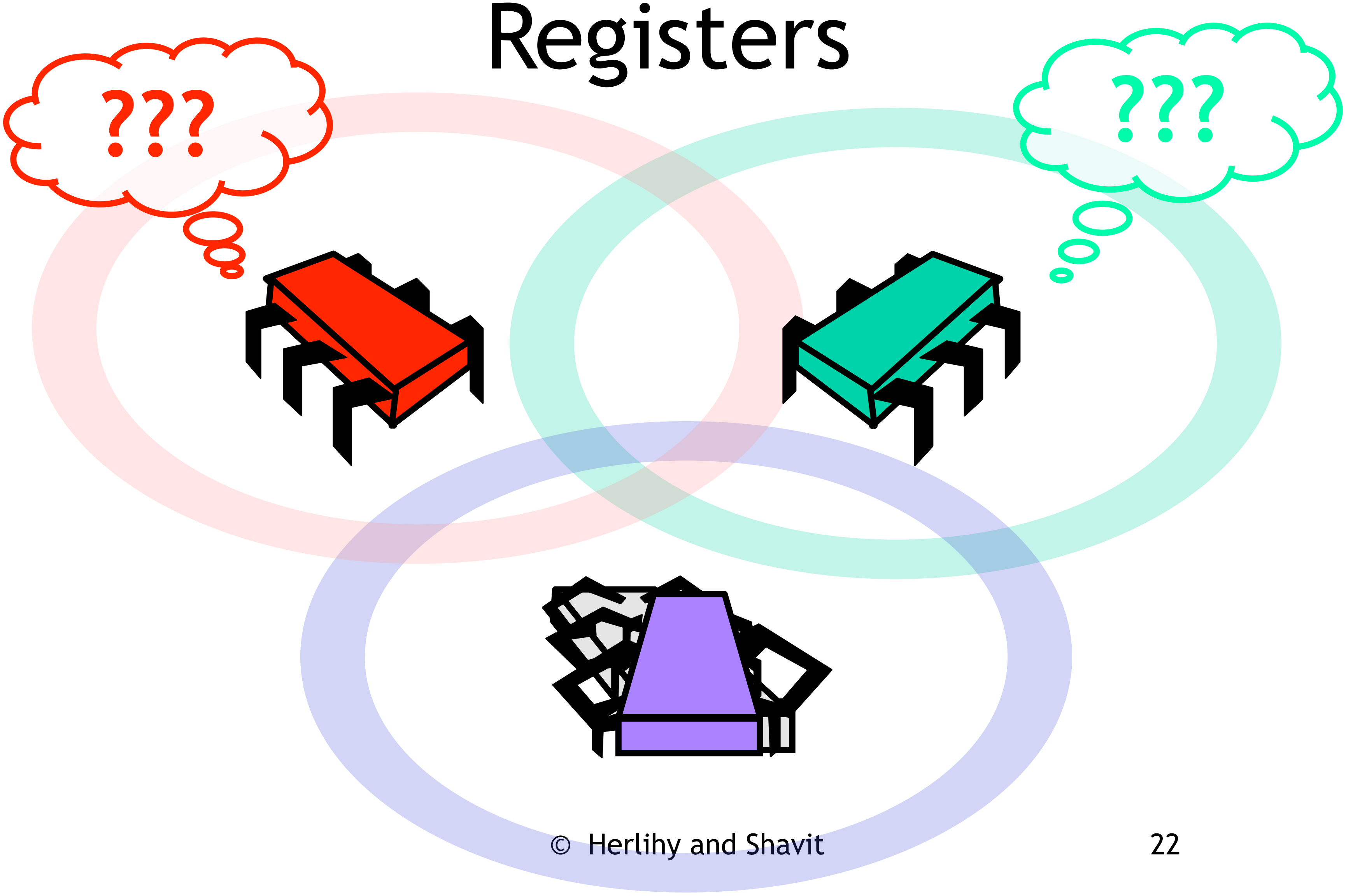
Formally: Consensus

Consistent: all threads decide the same value

Valid: the common decision value is some thread's input

Wait-free: each thread decides after a finite number of steps

No Wait-Free Consensus using Registers



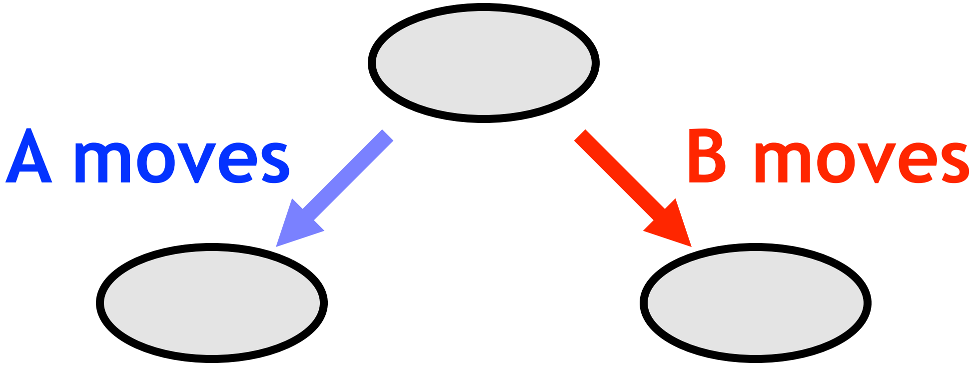
Formally

- Theorem [adapted from Fischer, Lynch, Paterson]: There is no wait-free implementation of n-thread consensus, n>1, from read-write registers even if only one thread can crash
- Implication: asynchronous computability fundamentally different from Turing computability

Proof Strategy

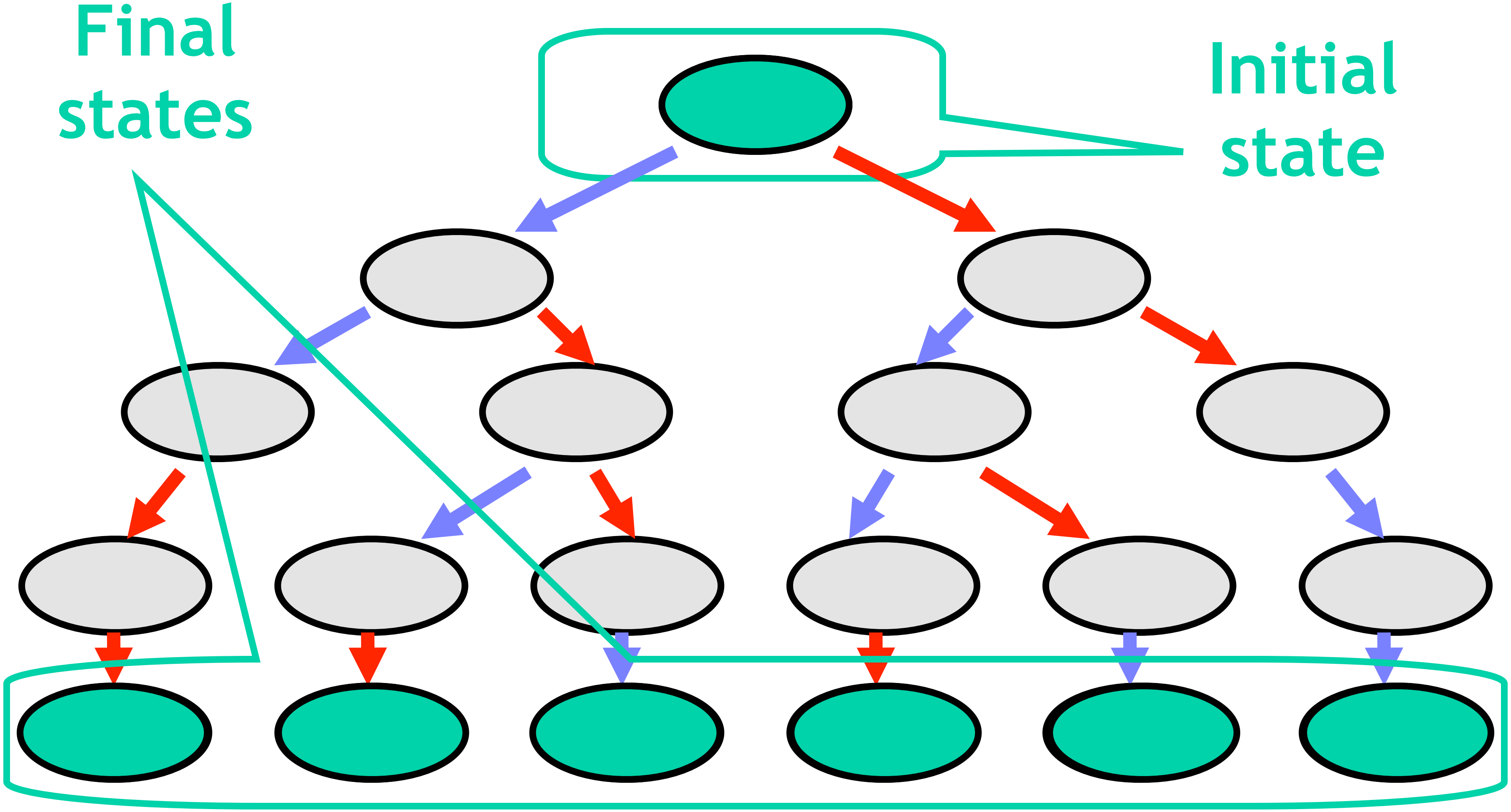
- Assume otherwise
- Reason about the properties of any such protocol
- Derive a contradiction
- Quod Erat Demonstrandum
- Suffices to prove for binary consensus and n=2

Wait-Free Computation



- Either A or B "moves"
- Moving means
  - Register read
  - Register write

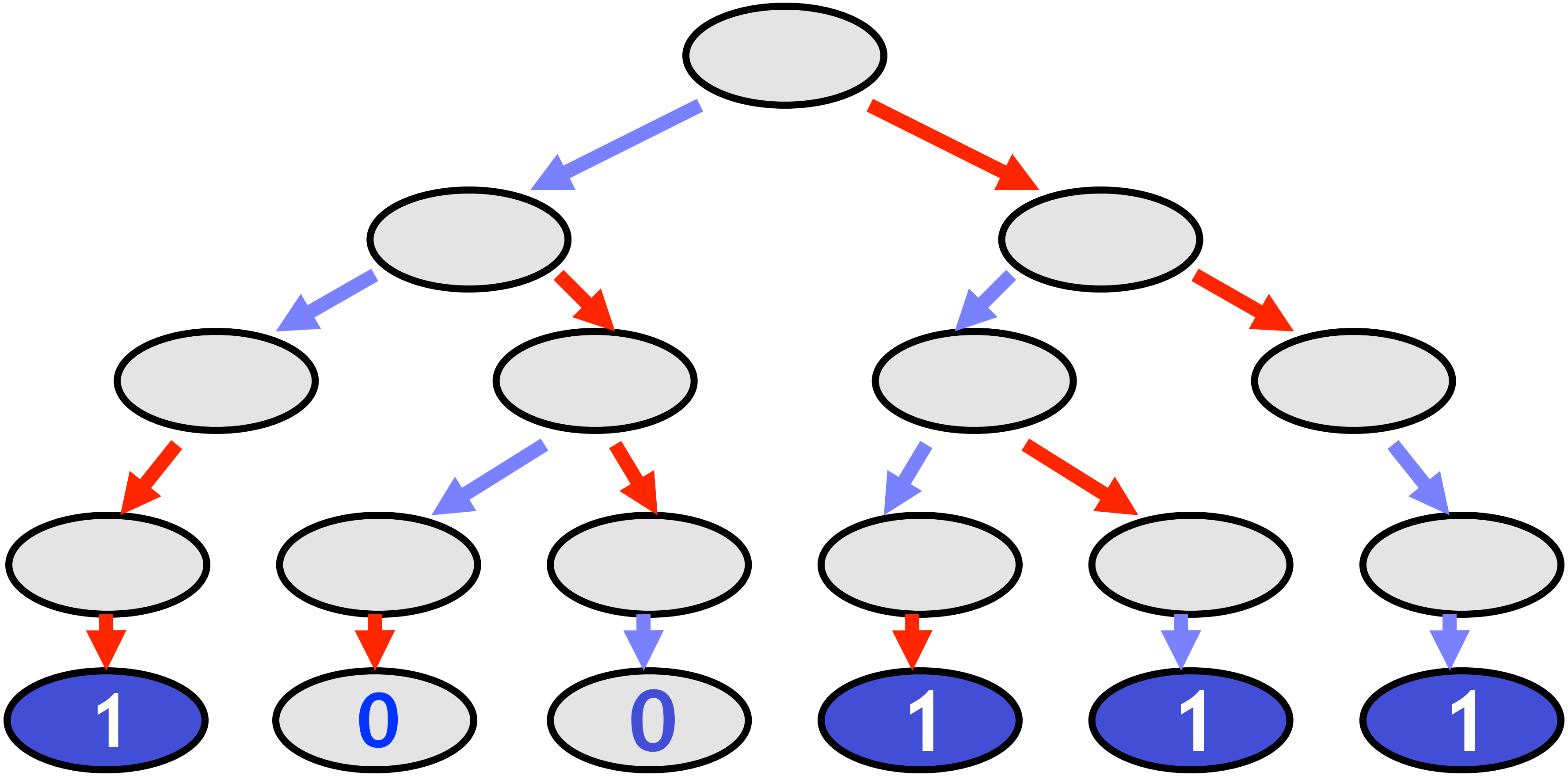
The Two-Move Tree



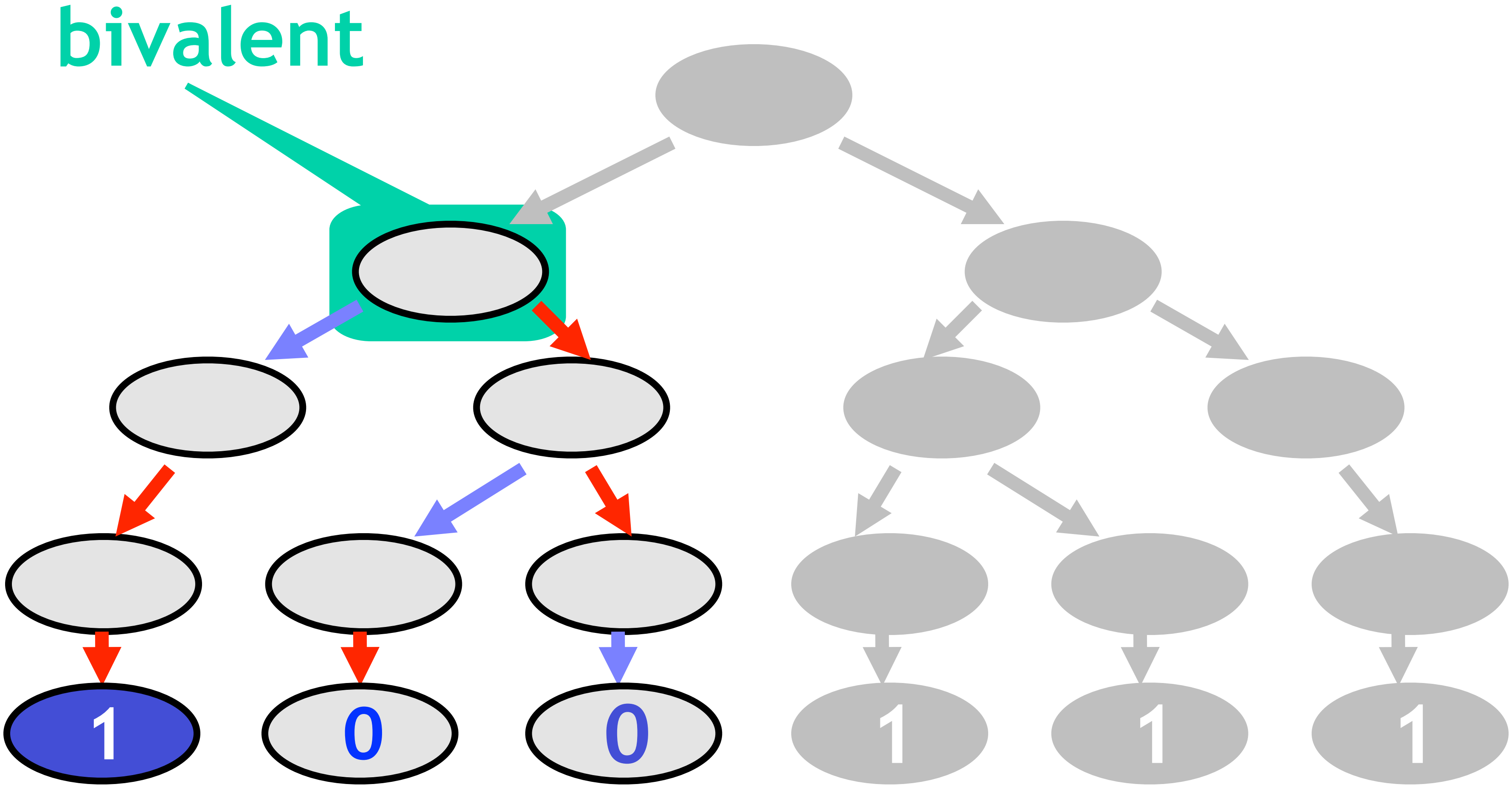
Final states

Initial state

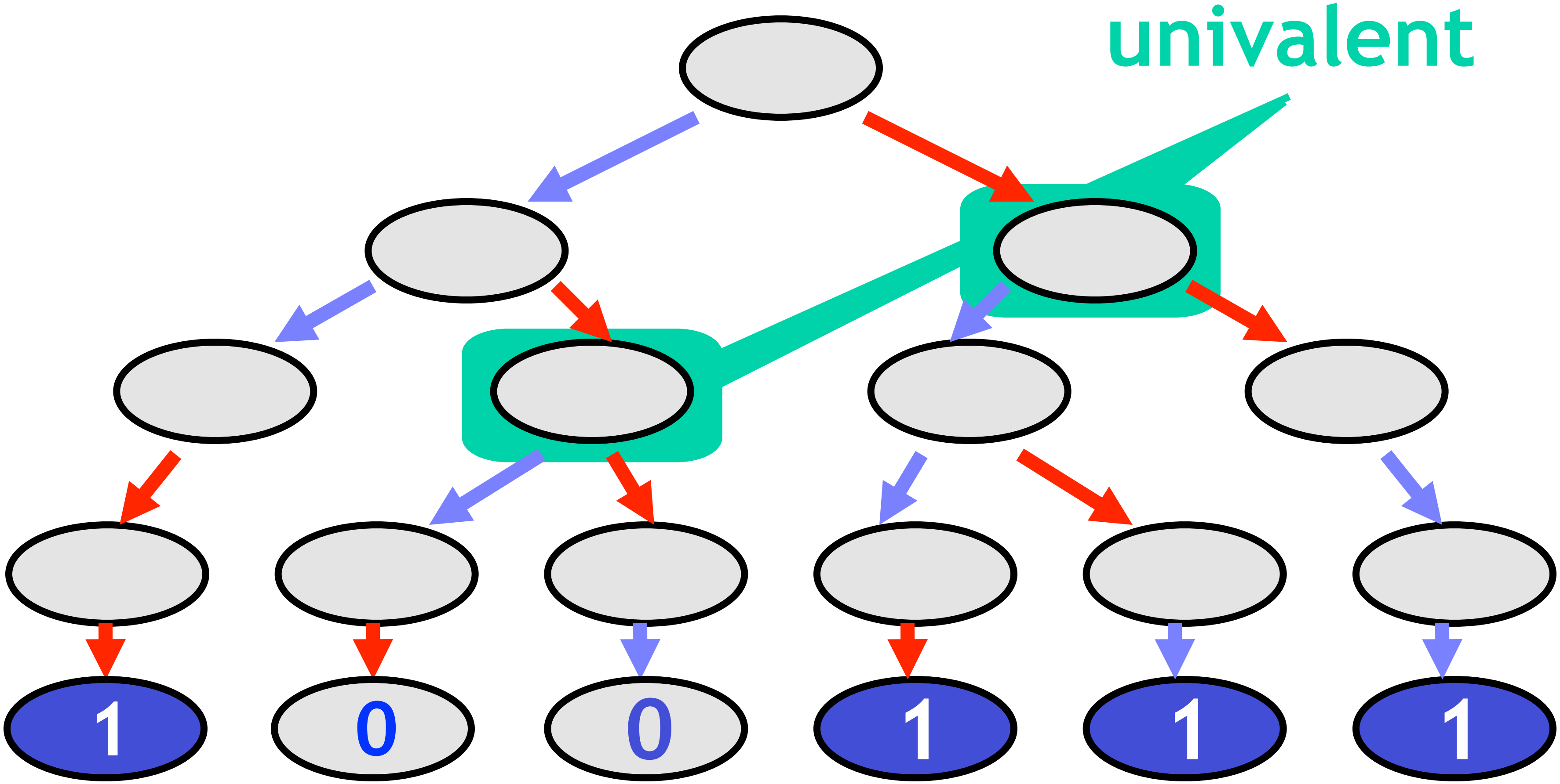
Decision Values



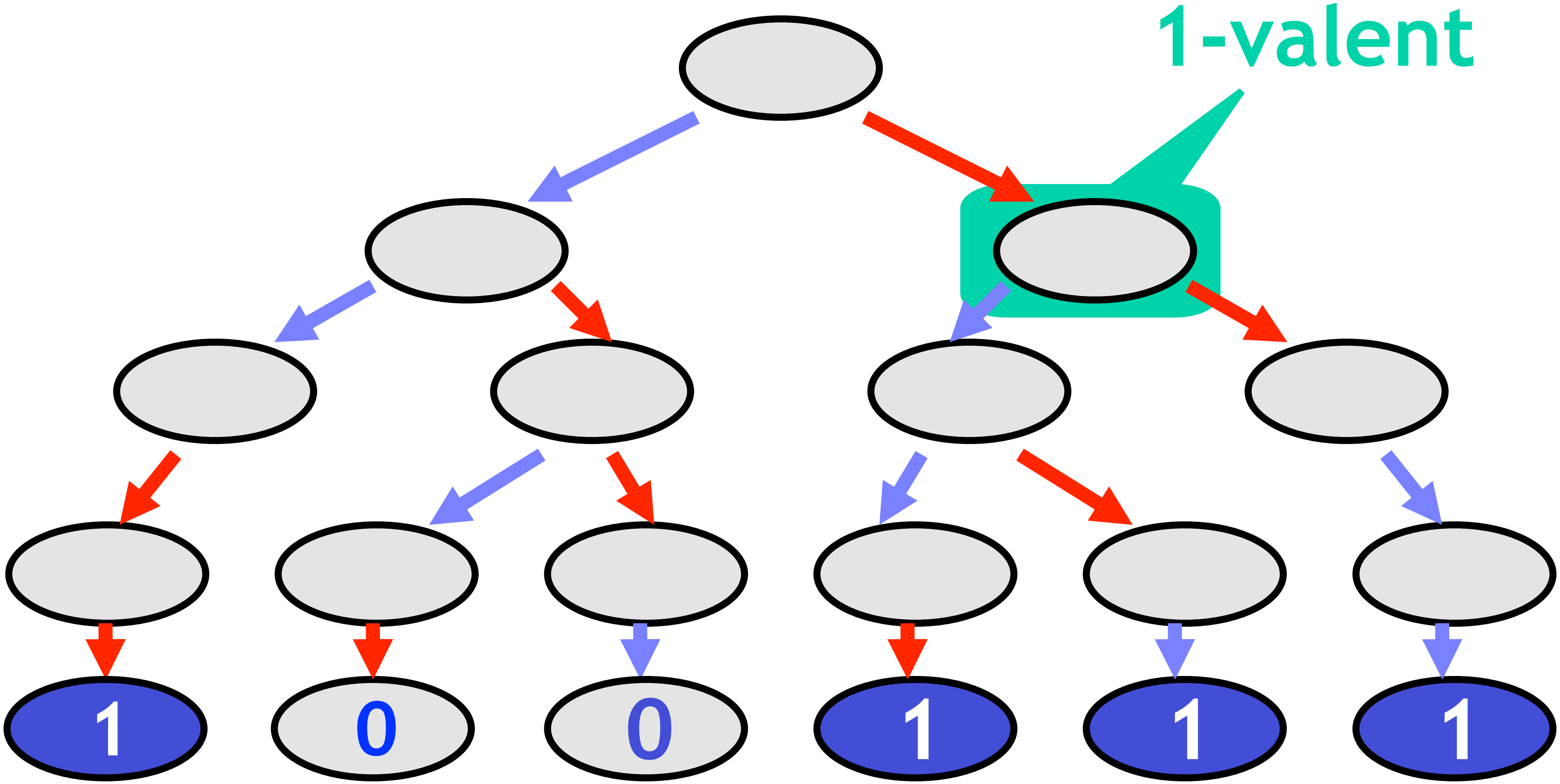
Bivalent: Both Possible



Univalent: Single Value Possible



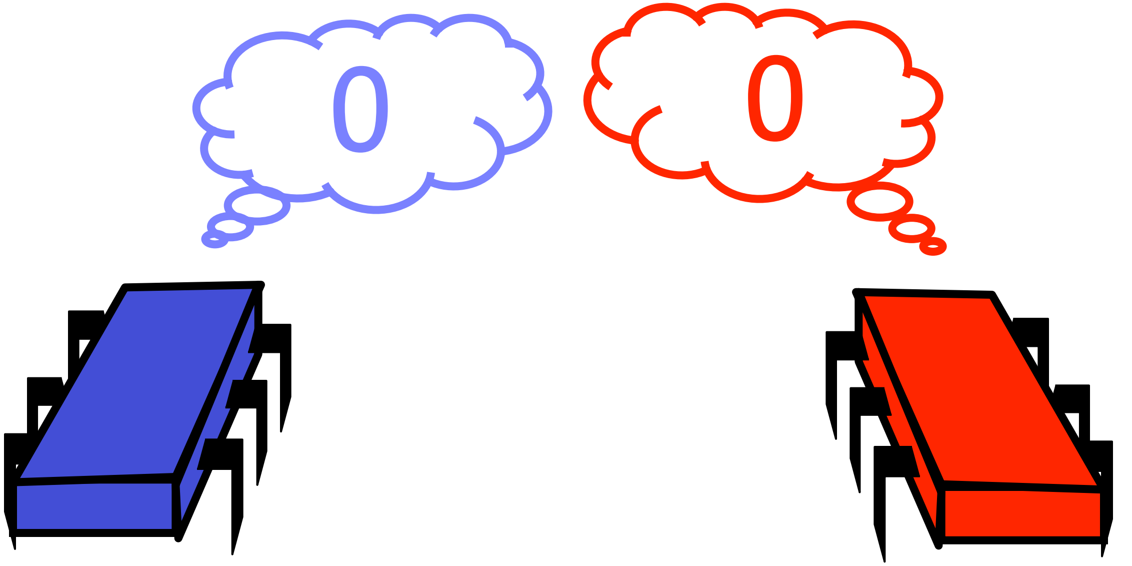
x-valent: x Only Possible Decision



Claim

- Some initial state is bivalent
- Outcome depends on
  - Chance
  - Behavior of the scheduler
- Lets prove this claim

Both Inputs 0

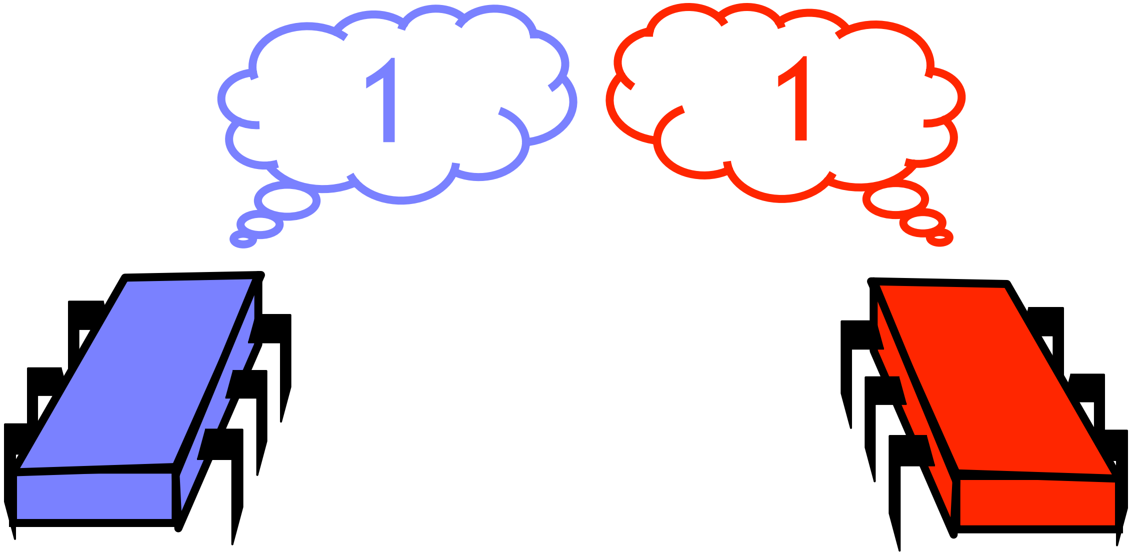


Univalent: all executions must decide 0

Both Inputs 0

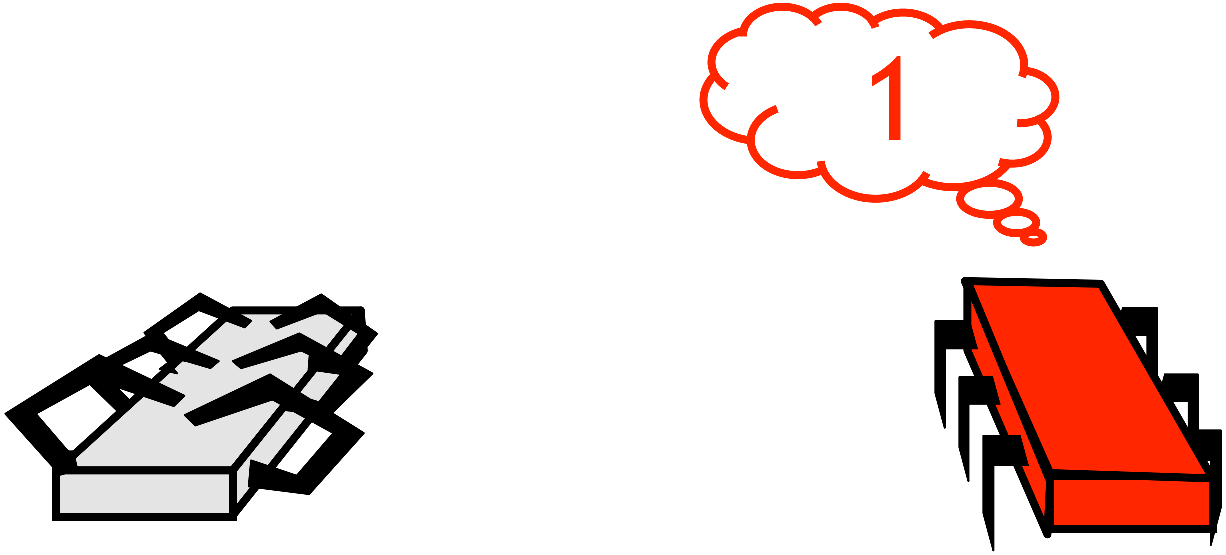
Including this solo execution by A

Both Inputs 1



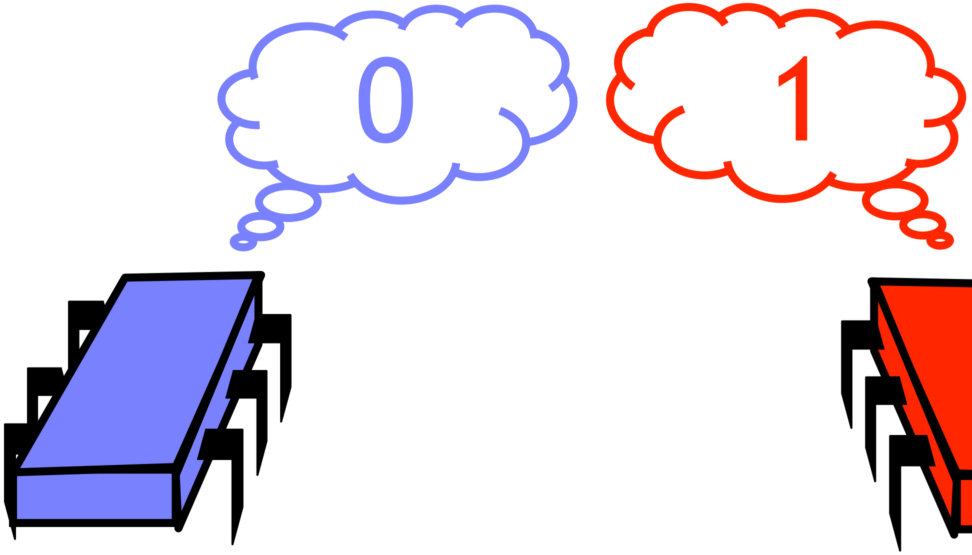
All executions must decide 1

Both Inputs 1



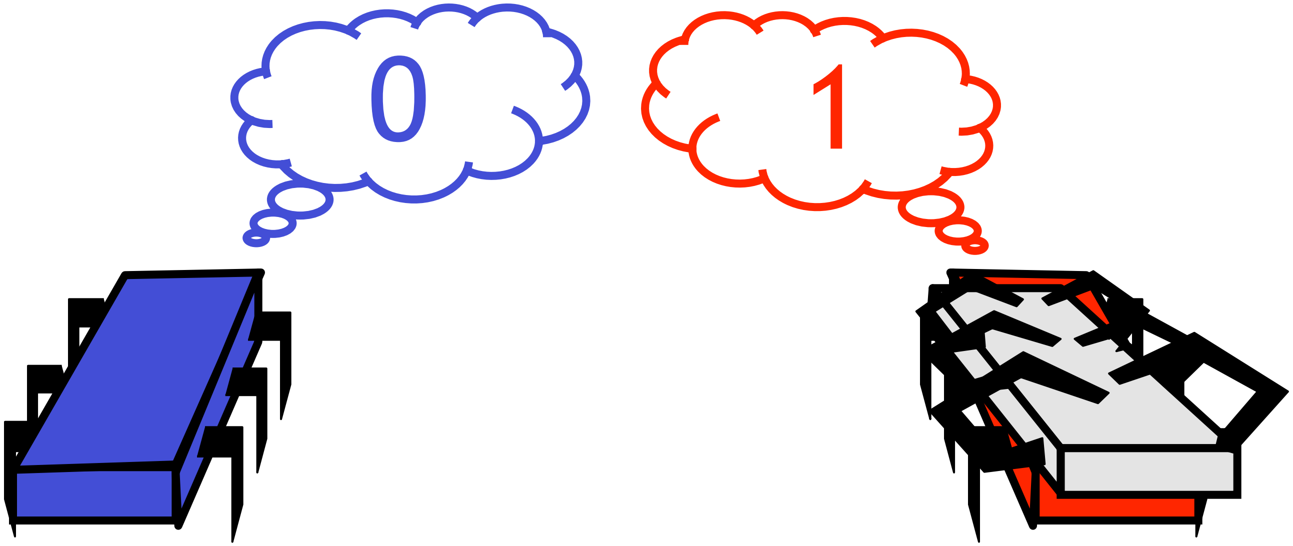
Including this solo execution by B

What if inputs differ?



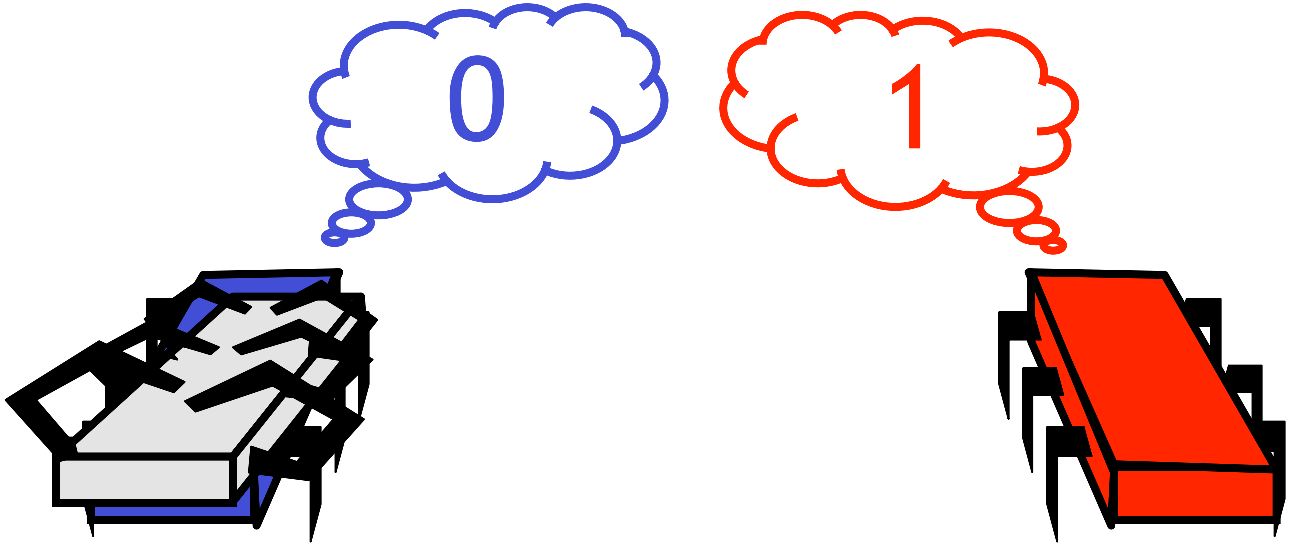
By Way of contradiction: If univalent all executions must decide on same value

The Possible Executions



Include the solo execution by A that decides 0

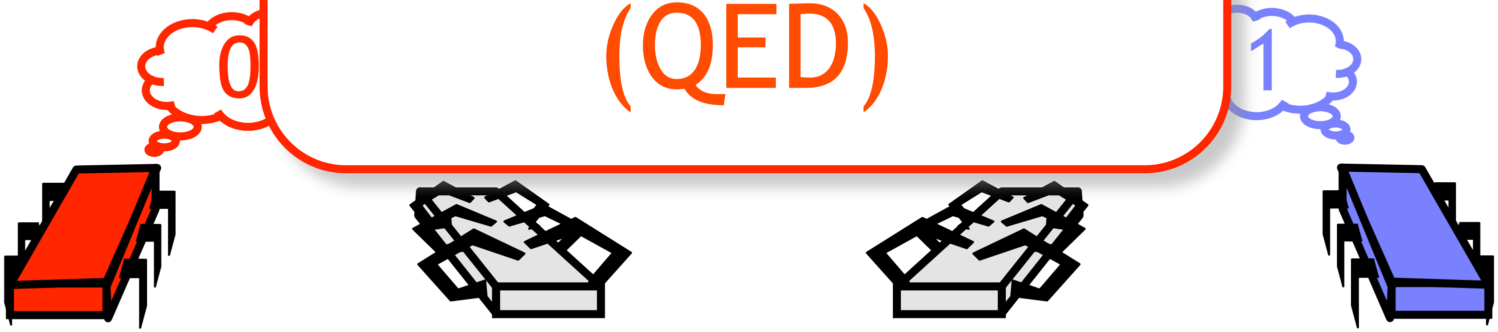
The Possible Executions



Also include the solo execution by B which we know decides 1

Possible Executions Include

How univalent is that?
(QED)

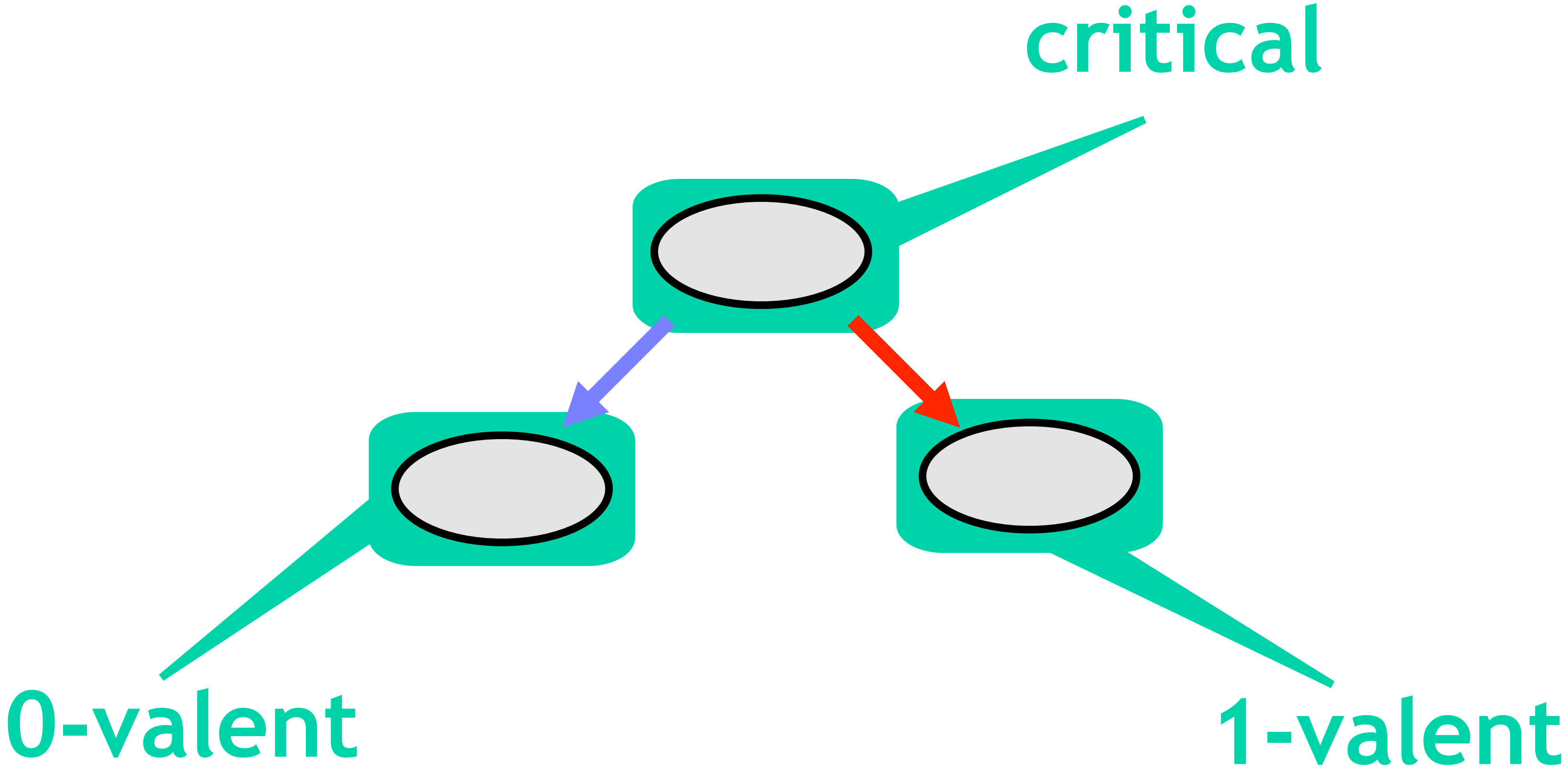


- Solo execution by A must decide 0
- Solo execution by B must decide 1

Summary So Far

- Wait-free computation is a tree
- Bivalent system states
  - Outcome not fixed
- Univalent states
  - Outcome is fixed
  - May not be “known” yet
- 1-Valent and 0-Valent states

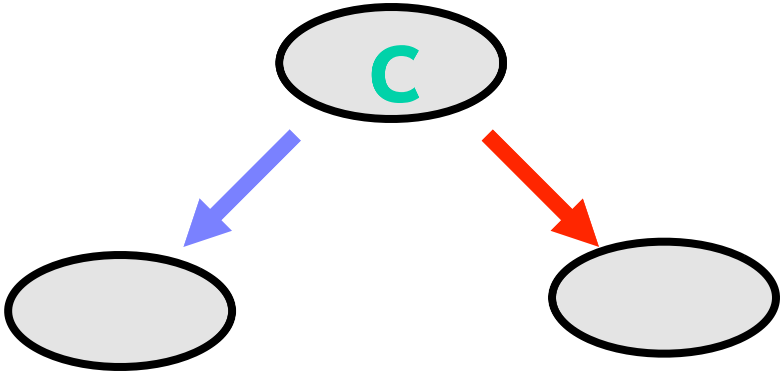
Critical States


critical

0-valent

1-valent

From a Critical State



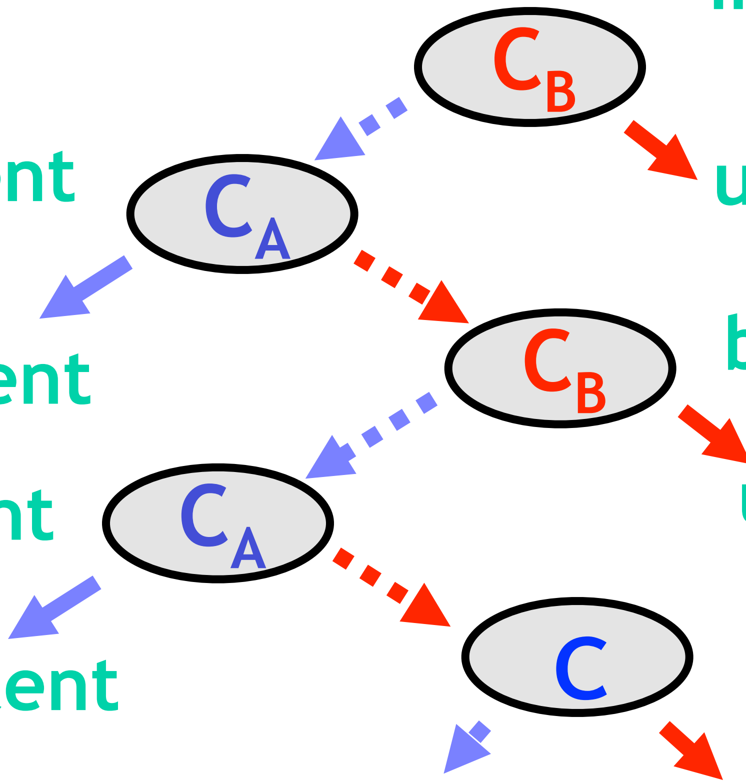
0-valent

1-valent

If A goes first, protocol decides 0

If B goes first, protocol decides 1

Reaching Critical State



initially bivalent

bivalent

univalent

univalent

bivalent

bivalent

univalent

univalent

0-valent

1-valent

© Herlihy and Shavit

Critical States

- Starting from a bivalent initial state
- The protocol can reach a critical state
  - Otherwise we could stay bivalent forever
  - And the protocol is not wait-free

Model Dependency

- So far, memory-independent!
- True for
  - Registers
  - Message-passing
  - Carrier pigeons
  - Any kind of asynchronous computation

What are the Threads Doing?

- Reads and/or writes
- To same/different registers

Completing the Proof

- Lets look at executions that:
  - Start from a critical state
  - Threads cause state to become univalent by reading or writing to same/different registers
  - End within a finite number of steps deciding either 0 or 1
- Show this leads to a contradiction

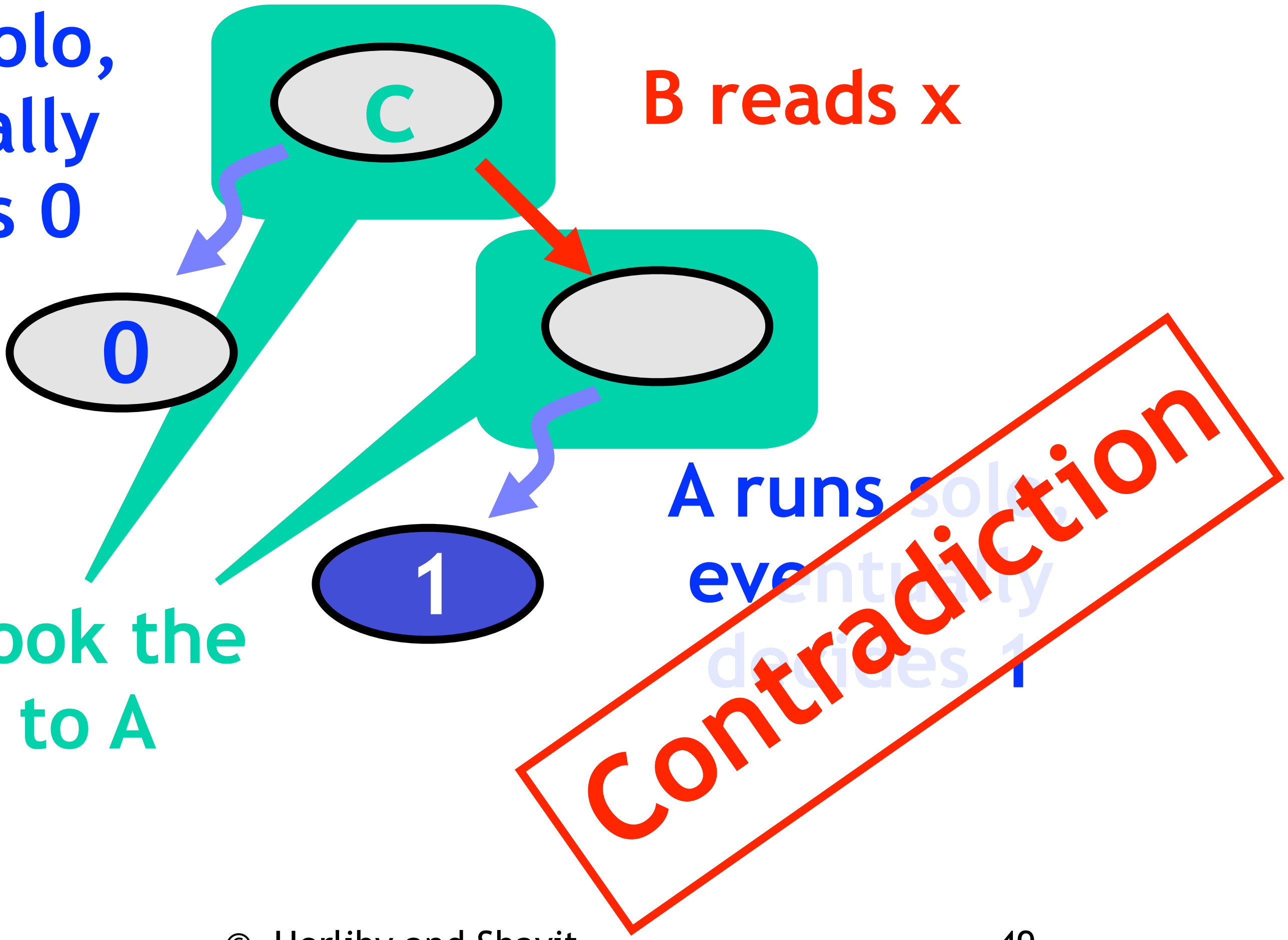
Possible Interactions

A reads x

A reads y

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	?	?	?	?
`y.read()`	?	?	?	?
`x.write()`	?	?	?	?
`y.write()`	?	?	?	?

Some Thread Reads



A runs solo, eventually decides 0

B reads x

States look the same to A

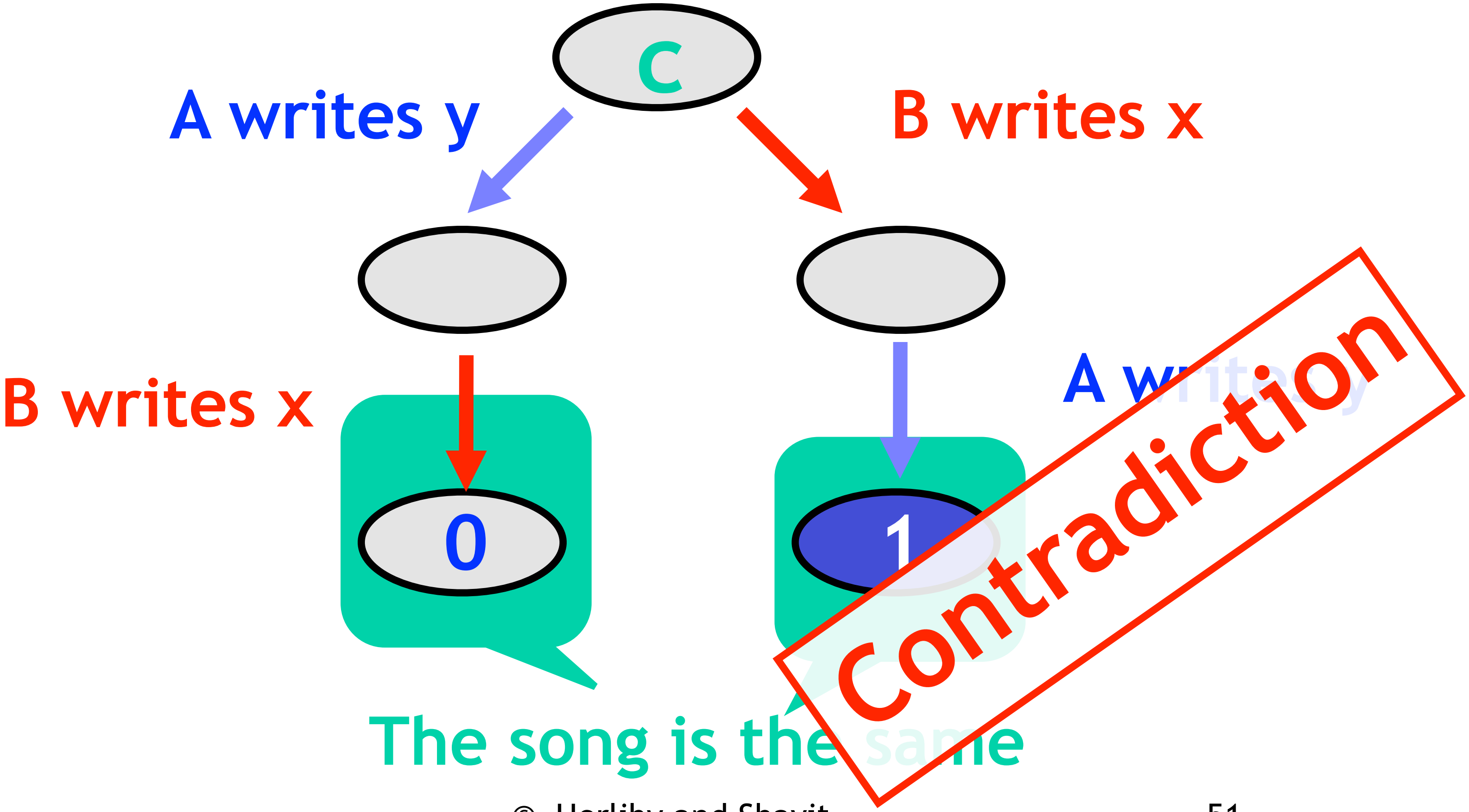
A runs solo, eventually decides 1

Contradiction

Possible Interactions

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	?	?
`y.write()`	no	no	?	?

Writing Distinct Registers



C

A writes y

B writes x

B writes x

0

A writes y

1

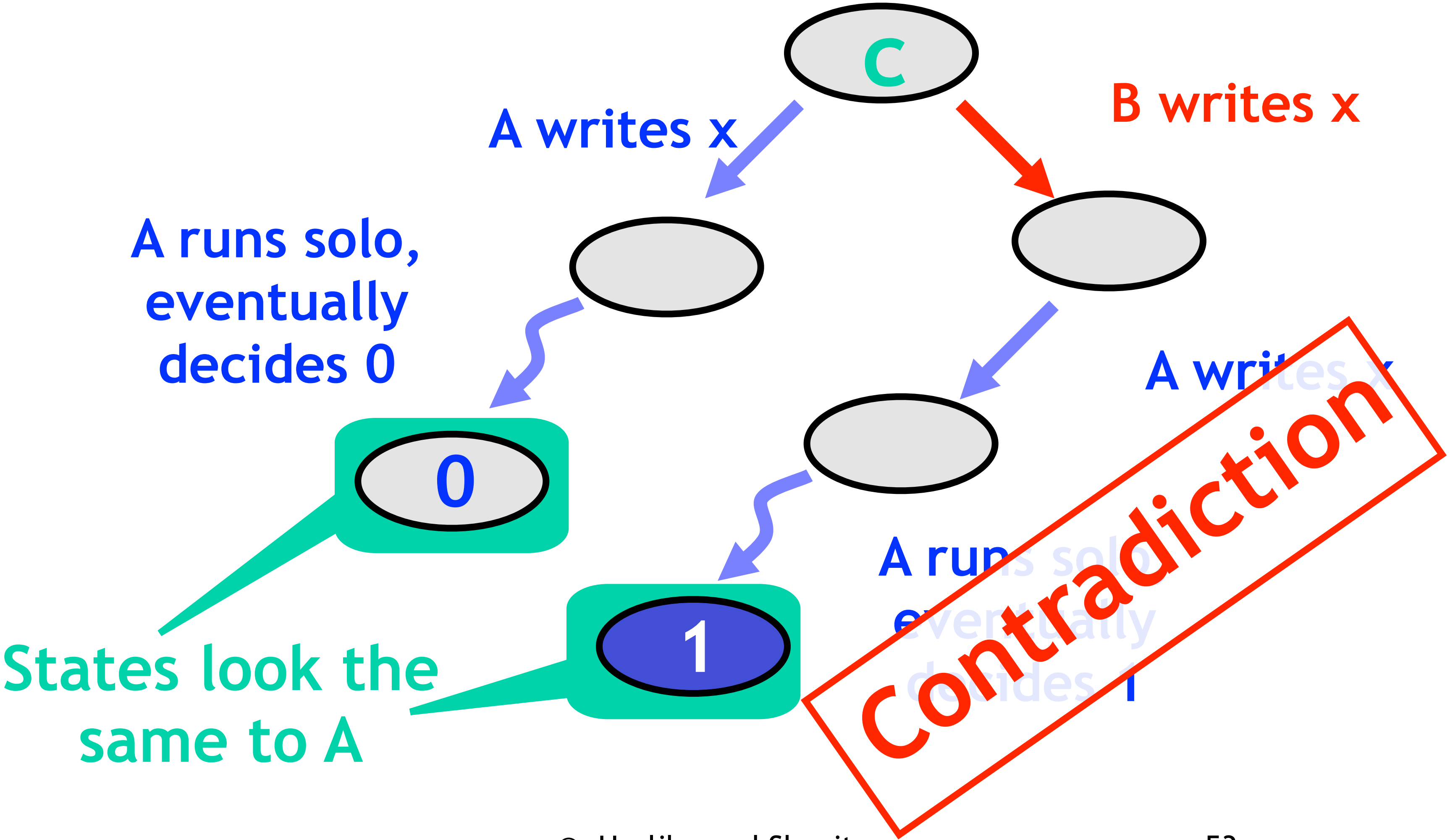
The song is the same

Contradiction

Possible Interactions

	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	?	no
`y.write()`	no	no	no	?

Writing Same Registers



B writes x

A writes x

A runs solo, eventually decides 0

A writes x

A runs solo, eventually decides 1

States look the same to A

Contradiction

That’s All, Folks!

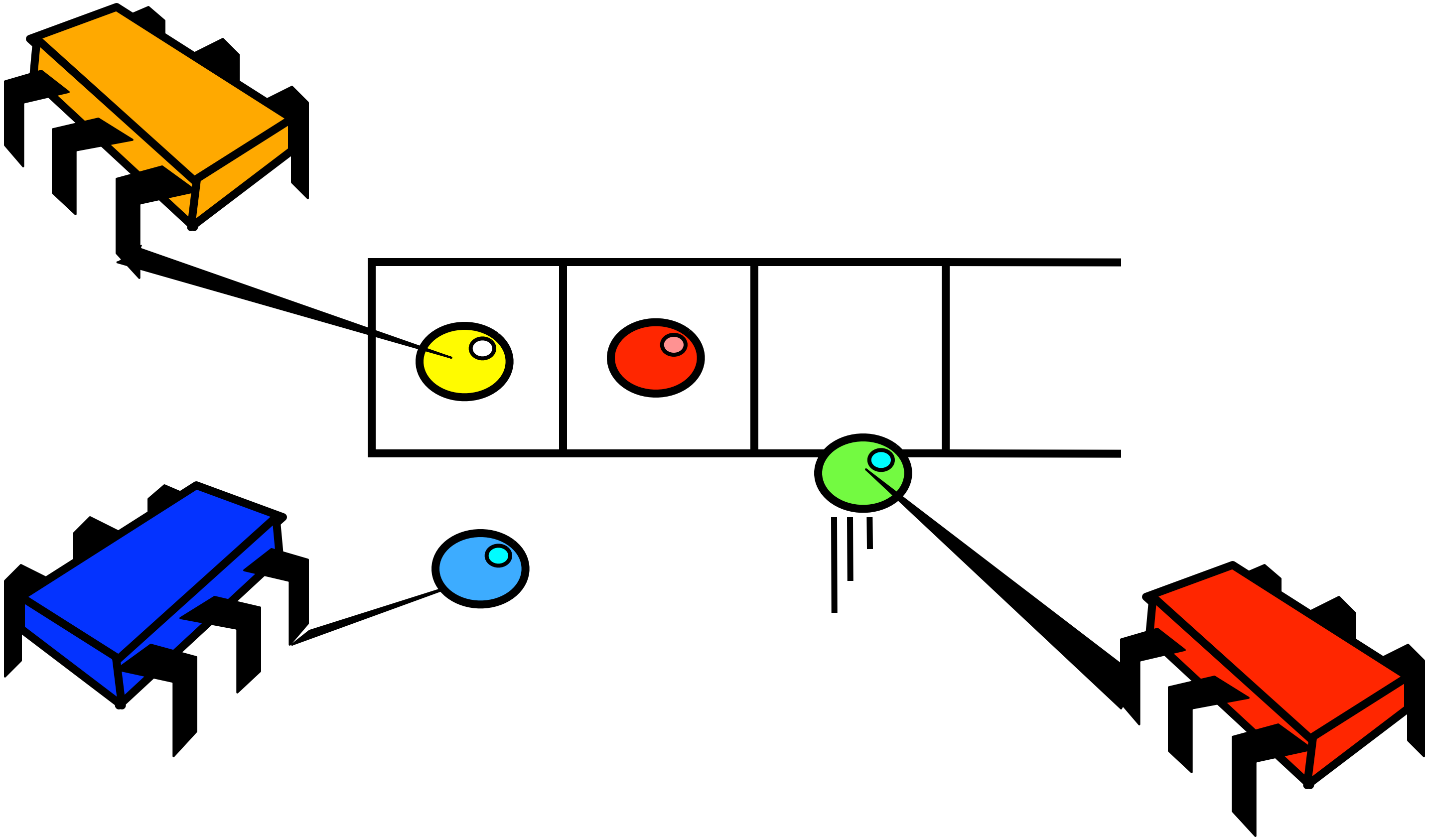
	`x.read()`	`y.read()`	`x.write()`	`y.write()`
`x.read()`	no	no	no	no
`y.read()`	no	no	no	no
`x.write()`	no	no	no	no
`y.write()`	no	no	no	no

QED

Recap: Atomic Registers Can't Do Consensus

- If protocol exists
  - It has a bivalent initial state
  - Leading to a critical state
- What's up with the critical state?
  - Case analysis for each pair of methods
  - As we showed, all lead to a contradiction

What Does Consensus have to do with Concurrent Objects?



Consensus Object

```
public interface Consensus {
 Object decide(Object value);
}
```

Concurrent Consensus Object

- We consider only one time objects: each thread can execute a method only once
- Linearizable to sequential consensus object in which
  - the thread who's input was decided on completed its method first

Java Jargon Watch

- Define Consensus protocol as an abstract class
- We implement some methods
- Leave you to do the rest …

Generic Consensus Protocol

```
abstract class ConsensusProtocol
implements Consensus {
 protected Object[] proposed =
  new Object[N];

 private void propose(Object value) {
  proposed[ThreadID.get()] = value;
 }

 abstract public Object
             decide(Object value);
 }}
```

Generic Consensus Protocol

```
abstract class ConsensusProtocol
implements Consensus {
 protected Object[] proposed =
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 private void propose(Object value) {
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 }

 abstract public Obje
                 deci

 }}
```

Each thread's proposed value

Generic Consensus Protocol

```
abstract class ConsensusProtocol
implements Consensus {
 protected Object[]
  new Object[N];

 private void propose(Object value) {
  proposed[ThreadID.get()] = value;
 }

 abstract public Object
                decide(object value);
 }}
```

Propose a value

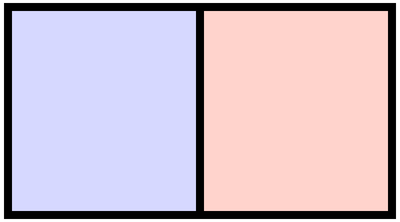
Generic Consensus Protocol

```
abstract class ConsensusProtocol

 private void propose(Object value) {
  proposed[ThreadID.get()] = value;
 }

 abstract public Object
             decide(object value);
 }}
```

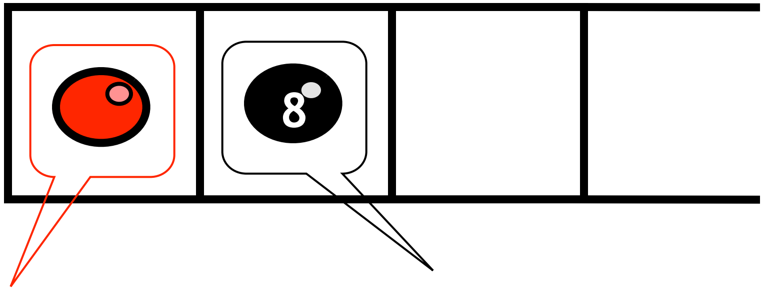
Decide a value: abstract method means subclass does the heavy lifting (real work)

Can FIFO Queue Implement Consensus?

FIFO Consensus



`propose` **array**

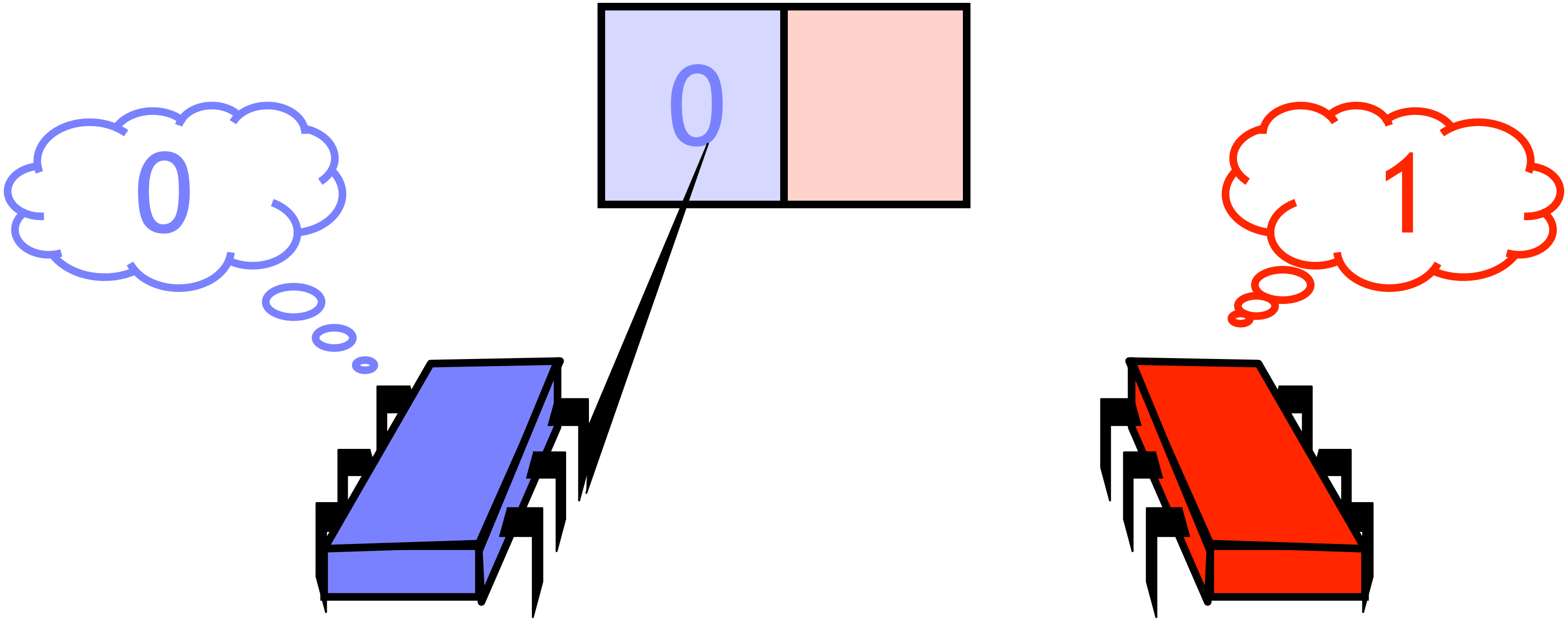


`FIFO Queue` **with red and black balls**

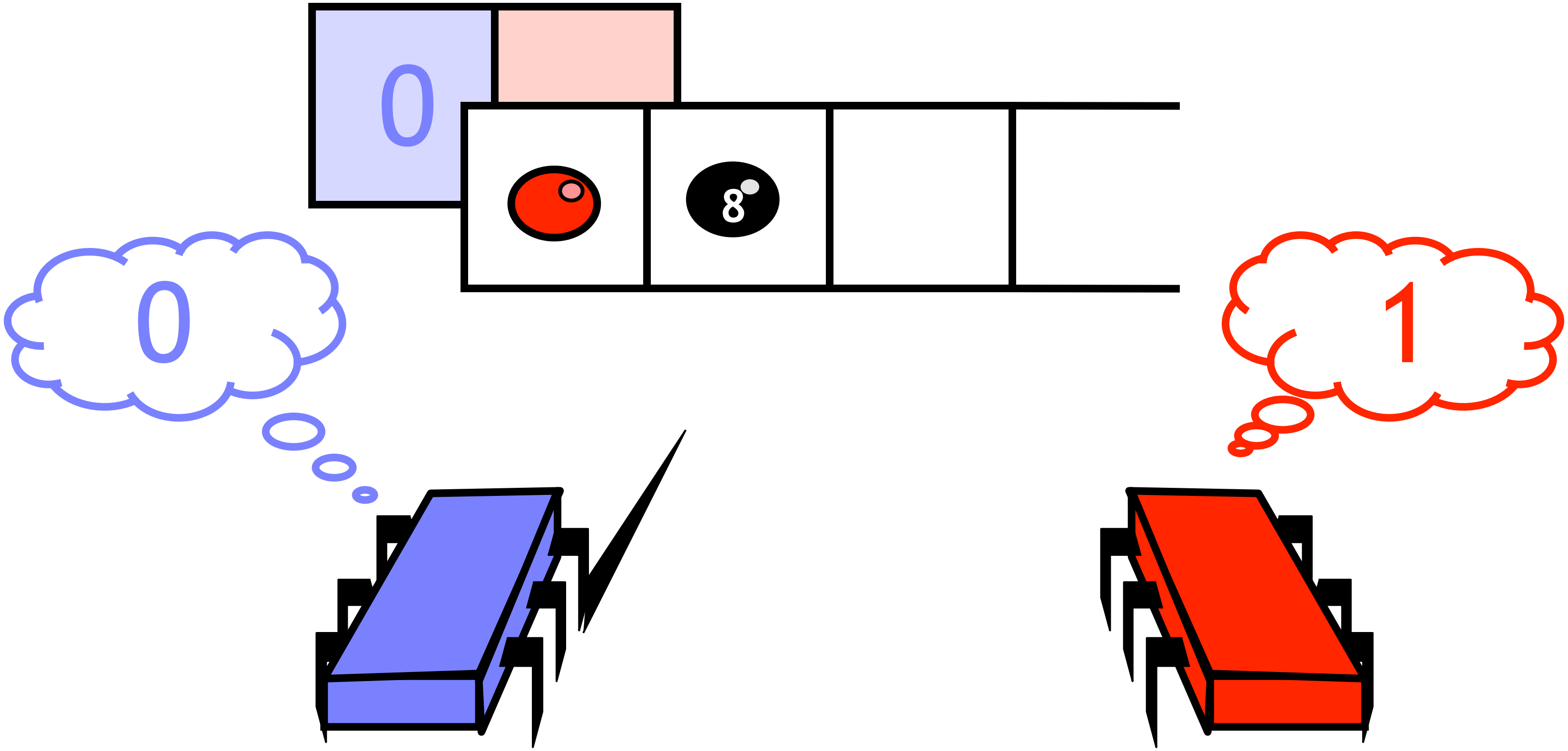
Coveted red ball

Dreaded black ball

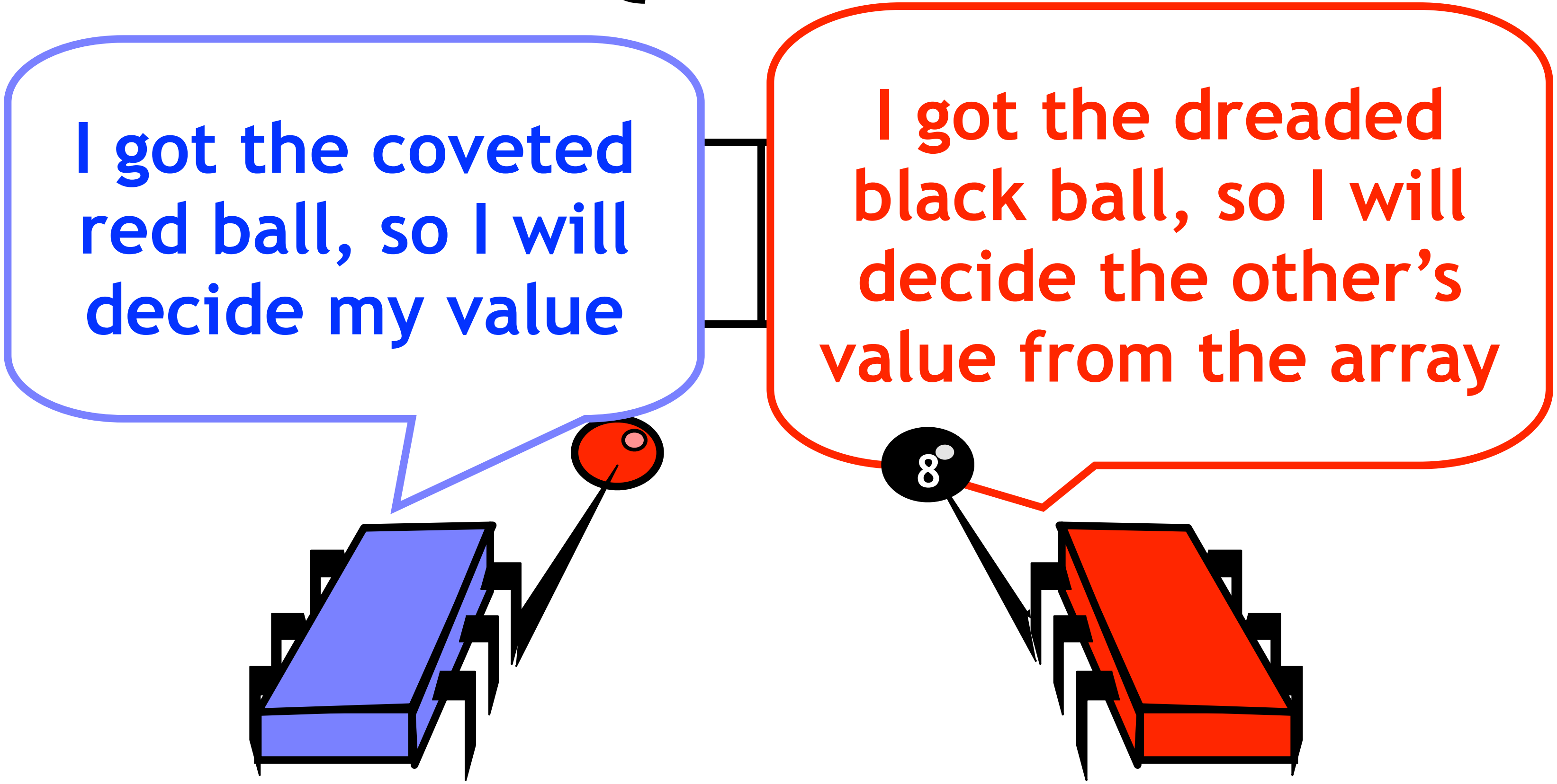
Protocol: Write Value to Array



Protocol: Take Next Item from Queue



Protocol: Take Next Item from Queue



I got the coveted red ball, so I will decide my value

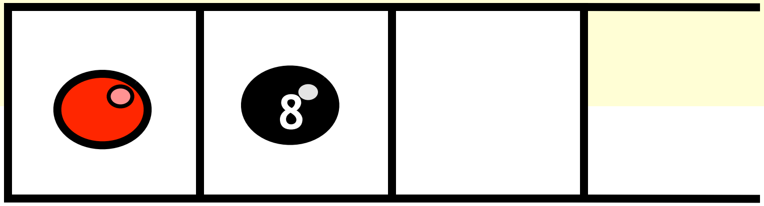
I got the dreaded black ball, so I will decide the other's value from the array

Consensus Using FIFO Queue

```
public class QueueConsensus
  extends ConsensusProtocol {
 private Queue queue;
 public QueueConsensus() {
  queue = new Queue();
  queue.enq(Ball.RED);
  queue.enq(Ball.BLACK);
 }
 …
}
```

Initialize Queue

```
public class QueueConsensus
  extends ConsensusProtocol {
 private Queue queue;
 public QueueConsensus() {
  queue = new Queue();
  queue.enq(Ball.RED);
  queue.enq(Ball.BLACK);
 }
 …
}
```



Who Won?

```
public class QueueConsensus
  extends ConsensusProtocol {
 private Queue queue;
 …
 public decide(object value) {
  propose(value);
  Ball ball = this.queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

Who Won?

```
public class QueueConsensus
  extends ConsensusProtocol {
 private Queue queue;
 …
 public decide(object value) {
  propose(value);
  Ball ball = this.queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-ij];
 }
}
```

Race to dequeue first queue item

Who Won?

```
public class QueueConsensus
  extends ConsensusProtocol {
 private Queue queue;
 …
 public decide(object value) {
  propose(value);
  Ball ball = this.queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

i = ThreadID.get();

I win if I was first

Who Won?

```
public class QueueConsensus
  extends ConsensusProtocol {
 private Queue queue;
 …
 public decide(object value) {
  propose(value);
  Ball ball = this.queue.deq();
  if (ball == Ball.RED)
   return proposed[i];
  else
   return proposed[1-i];
 }
}
```

Other thread wins if I was second

Why does this Work?

- If one thread gets the red ball
- Then the other gets the black ball
- Winner decides her own value
- Loser can find winner’s value in array
  - Because threads write to array
  - Before dequeueing from queue

Theorem

- We can solve 2-thread consensus using only
  - A two-dequeuer queue, and
  - Some atomic registers

Implications

- Given
  - A consensus protocol from queue and registers
- Assume there exists
  - A queue implementation from atomic registers
- Substitution yields:
  - A wait-free consensus protocol from atomic registers

contradiction

Corollary

- It is impossible to implement
  - a two-dequeuer wait-free FIFO queue
  - from read/write memory.

Consensus Numbers

- An object X has consensus number n
  - If it can be used to solve n-thread consensus
    - Taking any number of instances of X
    - together with atomic read/write registers
    - and implement n-thread consensus
  - But not (n+1)-thread consensus

Consensus Numbers

- Theorem
  - Atomic read/write registers have consensus number 1
- Theorem
  - Multi-dequeuer FIFO queues have consensus number at least 2

Consensus Numbers Measure Synchronization Power

- Theorem
  - If you can implement X from Y
  - And X has consensus number c
  - Then Y has consensus number at least c

Synchronization Speed Limit

- Conversely
  - If X has consensus number c
  - And Y has consensus number d < c
  - Then there is no way to construct a wait-free implementation of X by Y
- This theorem will be very useful
  - Unforeseen practical implications!

Theoretical Caveat: Certain weird exceptions exist

Homework

- What is the consensus number of a wait-free FIFO queue with methods:
  - enq(o): enqueue object o
  - deq(): dequeue first object
  - peek(): get a copy of first object

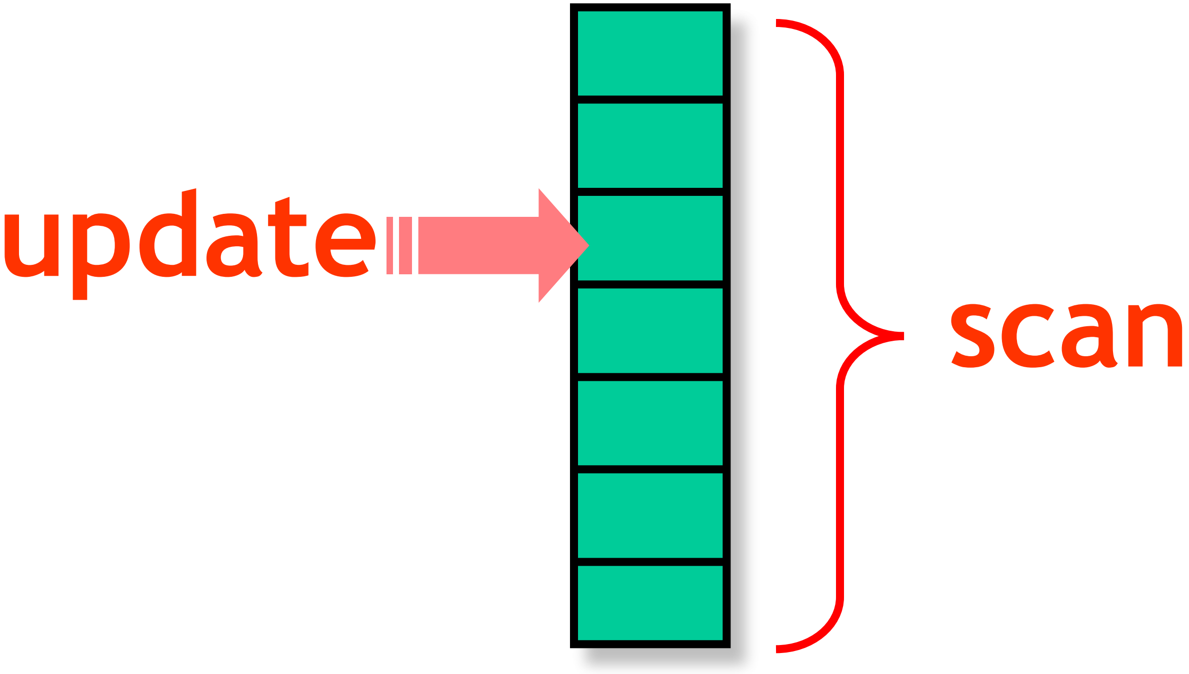
New Grand Challenge

- Consider:
  - Write multiple array elements atomically
  - Scan any array elements
- Call this problem multiple assignment

Multiple Assignment Theorem

- Atomic registers cannot implement multiple assignment
- Weird or what?
  - Single location write/multiple location read OK (= Atomic Snapshot)
  - Multi location write/single location read impossible

Atomic Snapshot



Atomic Snapshot

- Array of MRSW atomic registers
- Take instantaneous snapshot of all
- Generalizes to MRMW registers …

Snapshot Interface

```
public interface Snapshot {
  public int update(int v);
  public int[] scan();
}
```

Snapshot Interface

Thread i writes v to its register

```
public interface Snapshot {
  public int update(int v);
  public int[] scan();
}
```

Snapshot Interface

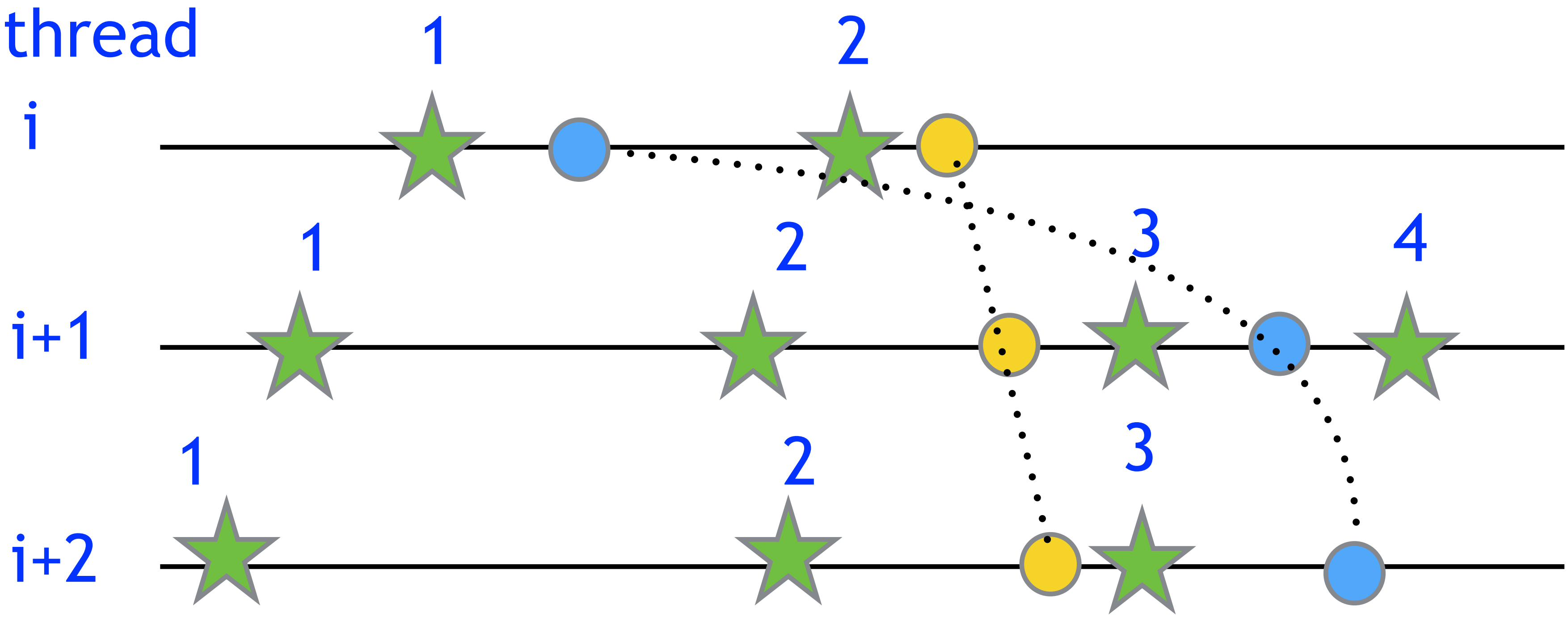
Instantaneous snapshot of all theads’ registers

```
public interface Snapshot {
  public int update(int v);
  public int[] scan();
}
```

Atomic Snapshot

- Collect
  - Read values one at a time
- Problem
  - Incompatible concurrent collects
  - Result not linearizable

Example: Atomic Snapshot MRMW



○ read

☆ update

A: 1, 3, 3

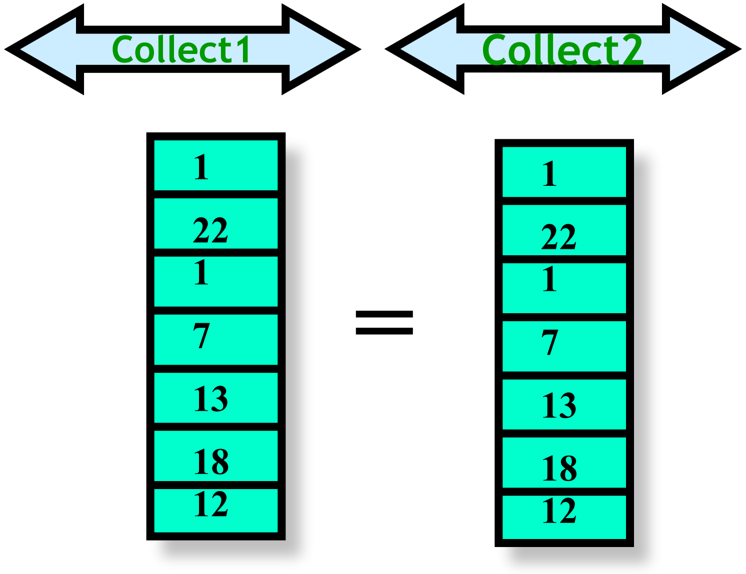
B: 2, 2, 2

Clean Collects

- Clean Collect
  - Collect during which nothing changed
  - Can we make it happen?
  - Can we detect it?

Simple Snapshot

- Put increasing labels on each entry
- Collect twice
- If both agree,
  - We’re done
- Otherwise,
  - Try again



Simple Snapshot: Update

```
public class SimpleSnapshot implements Snapshot {
  private AtomicMRSWRegister[] register;

  public void update(int value) {
    int i = Thread.myIndex();
       LabeledValue oldValue = register[i].read();
    LabeledValue newValue =
     new LabeledValue(oldValue.label+1, value);
    register[i].write(newValue);
  }
```

Simple Snapshot: Update

```
public class SimpleSnapshot implements Snapshot {
  private AtomicMRSWRegister[] register;

  public void update(int value) {
    int i = Thread.myIndex();
    LabeledValue oldValue = register[i].read();
    LabeledValue newValue =
     new LabeledValue(oldValue.label+1, value);
    register[i].write(newValue);
  }
```

One single-writer register per thread

Simple Snapshot: Update

```
public class SimpleSnapshot implements Snapshot {
  private AtomicMRSWRegister[] register;

  public void update(int value) {
    int i = Thread.myIndex();
    LabeledValue oldValue = register[i].read();
    LabeledValue newValue =
     new LabeledValue(oldValue.label+1, value);
    register[i].write(newValue);
  }
```

Write each time with higher label

Simple Snapshot: Collect

```
private LabeledValue[] collect() {
 LabeledValue[] copy =
  new LabeledValue[n];
 for (int j = 0; j < n; j++)
  copy[j] = this.register[j].read();
 return copy;
}
```

Simple Snapshot

```
private LabeledValue[] collect() {
 LabeledValue[] copy =
  new LabeledValue[n];
 for (int j = 0; j < n; j++)
  copy[j] = this.register[j].read();
 return copy;
}
```

Just read each register into array

Simple Snapshot: Scan

```
public int[] scan() {
  LabeledValue[] oldCopy, newCopy;
  oldCopy = collect();
  collect: while (true) {
    newCopy = collect();
    if (!equals(oldCopy, newCopy)) {
      oldCopy = newCopy;
      continue collect;
     }
    return getValues(newCopy);
  }
}
```

Simple Snapshot: Scan

```
public int[] scan() {
  LabeledValue[] oldCopy, newCopy;
  oldCopy = collect();
  collect: while (true) {
    newCopy = collect();
    if (!equals(oldCopy, newCopy)) {
      oldCopy = newCopy;
      continue collect;
    }
    return getValues(newCopy);
  }
}
```

Collect once

Simple Snapshot: Scan

```
public int[] scan() {
  LabeledValue[] oldCopy, newCopy;
  oldCopy = collect();
  collect: while (true) {
    newCopy = collect();
    if (!equals(oldCopy, newCopy)) {
      oldCopy = newCopy;
      continue collect;
    }
    return getValues(newCopy);
  }
}
```

Collect once

Collect twice

Simple Snapshot: Scan

```
public int[] scan() {
  LabeledValue[] oldCopy, newCopy;
  oldCopy = collect();
  collect: while (true) {
    newCopy = collect();
    if (!equals(oldCopy, newCopy)) {
      oldCopy = newCopy;
      continue collect;
    }
    return getValues(newCopy);
  }
}
```

Collect once

Collect twice

On mismatch, try again

(1)

Simple Snapshot: Scan

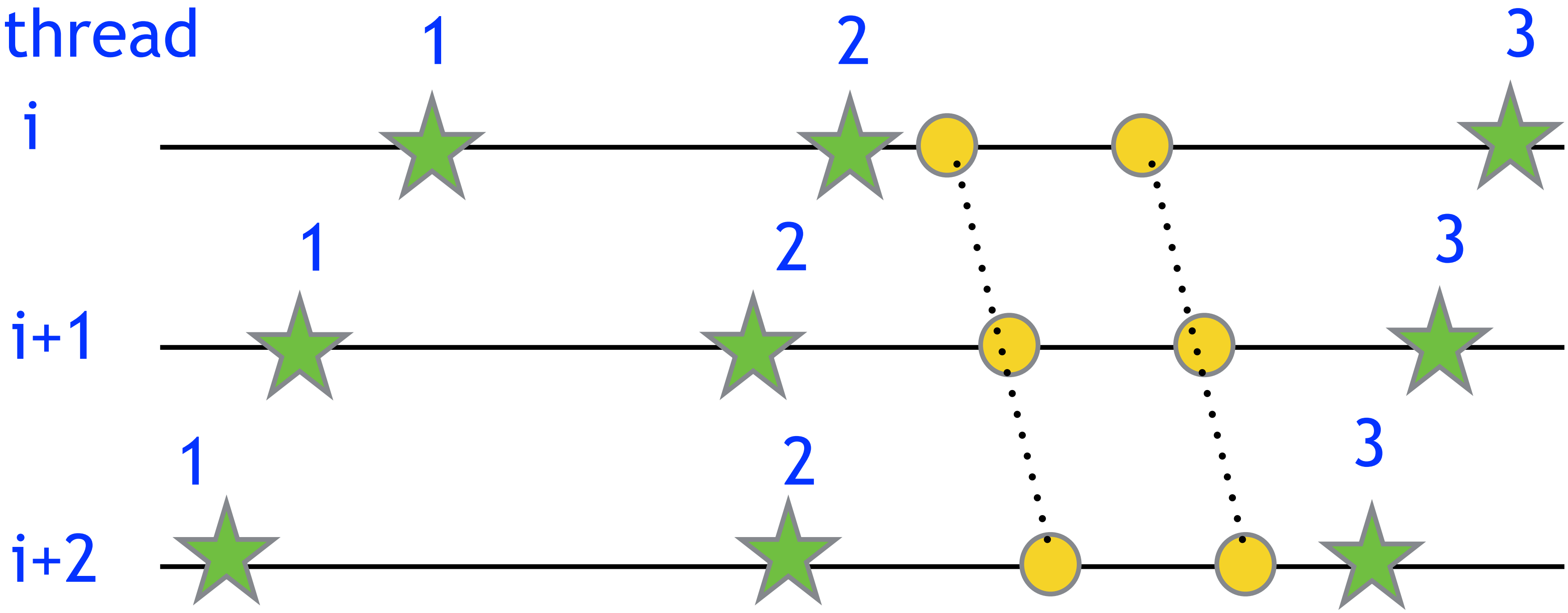
```
public int[] scan() {
  LabeledValue[] oldCopy, newCopy;
  oldCopy = collect();
  collect: while (true) {
    newCopy = collect();
    if (!equals(oldCopy, newCopy)) {
      oldCopy = newCopy;
      continue collect;
    }
    return getValues(newCopy);
  }
}
```

Collect once

Collect twice

On match, return values

Example



○ read

☆ update

B: 2, 2, 2

B: 2, 2, 2

Simple Snapshot

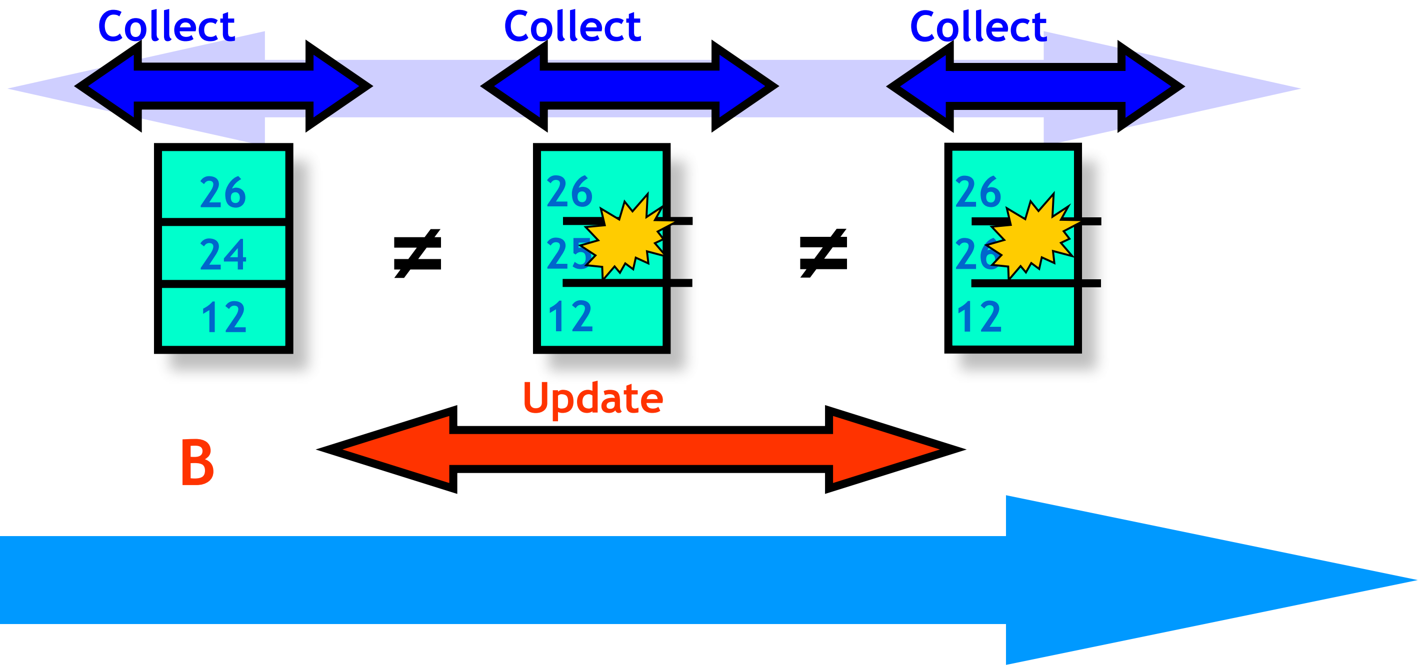
- Linearizable
- Update is wait-free
  - No unbounded loops
- But Scan can starve
  - If interrupted by concurrent update

Wait-Free Snapshot

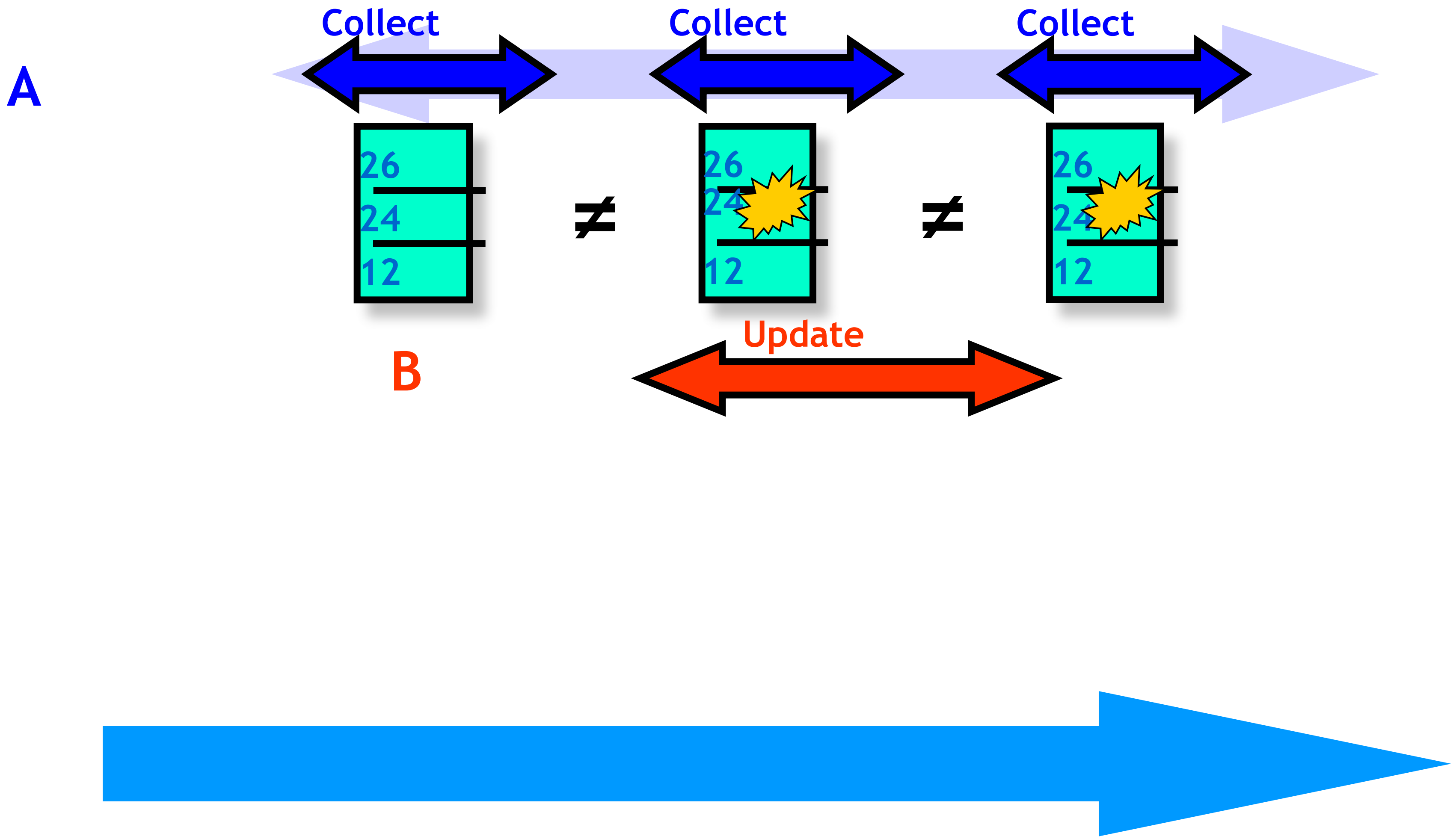
- Add a scan before every update
- Write resulting snapshot together with update value
- If scan is continuously interrupted by updates, scan can take the update's snapshot

Wait-free Snapshot

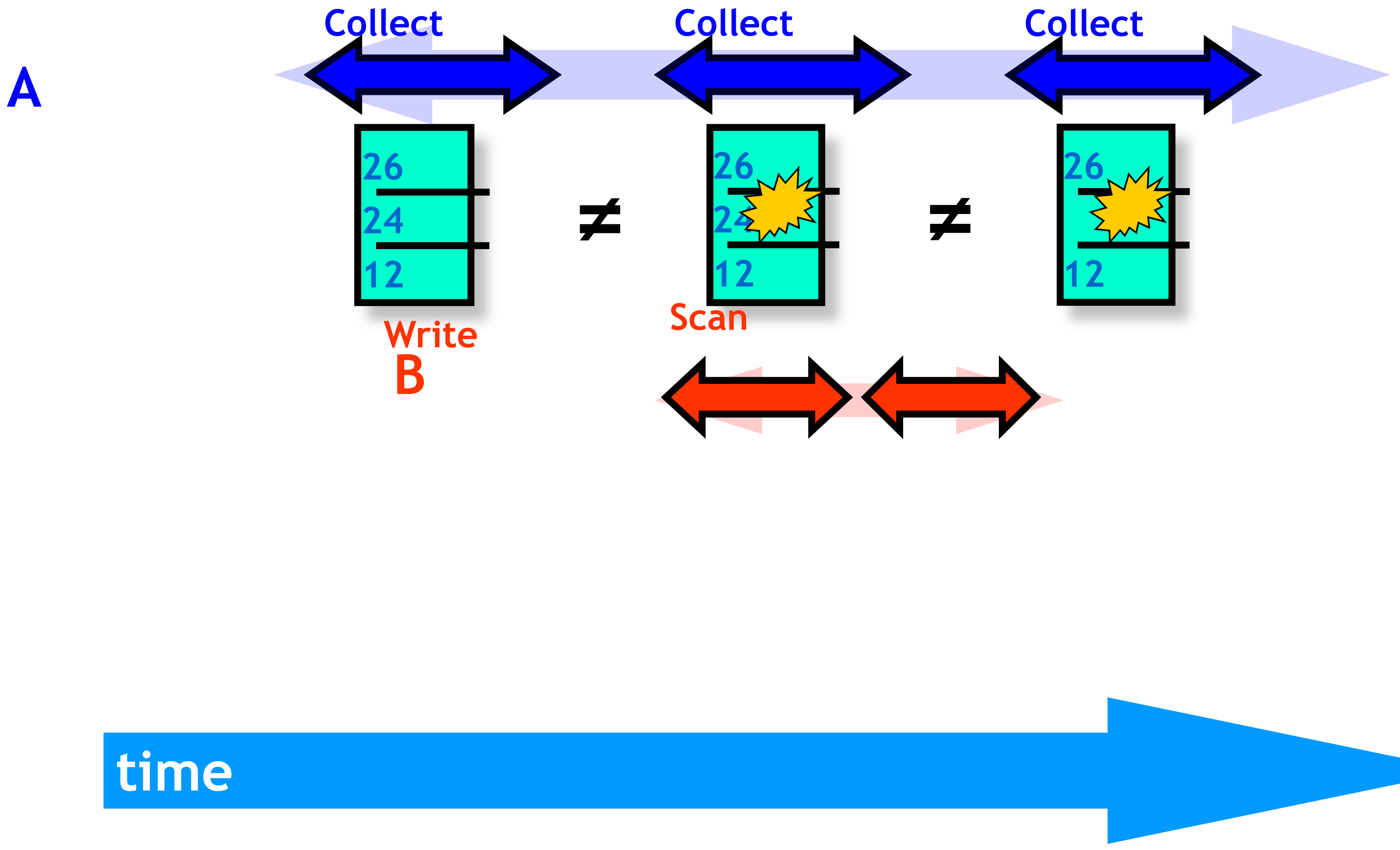
If A's scan observes that B moved <u>twice</u>, then B completed an update while A's scan was in progress



Wait-free Snapshot

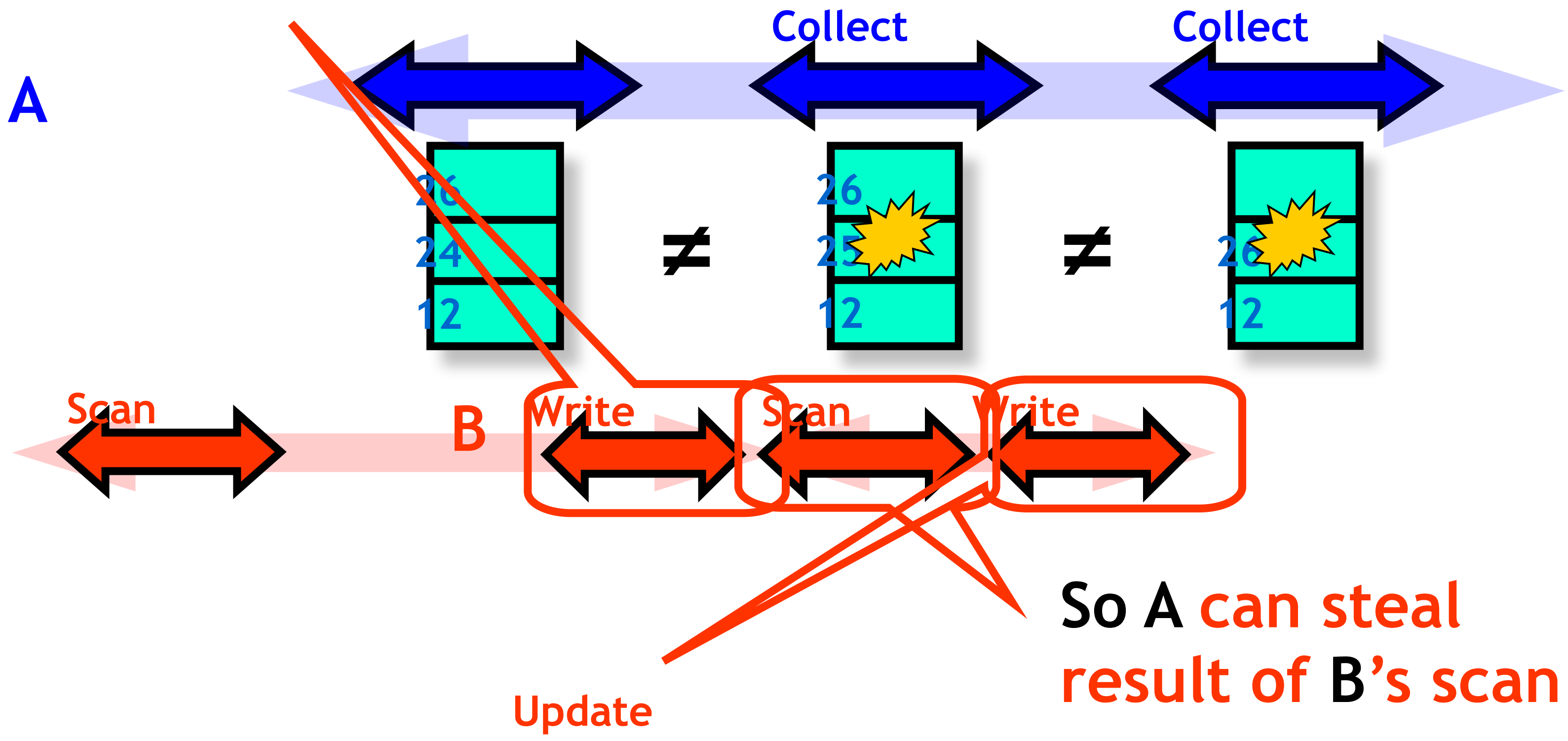


Wait-free Snapshot



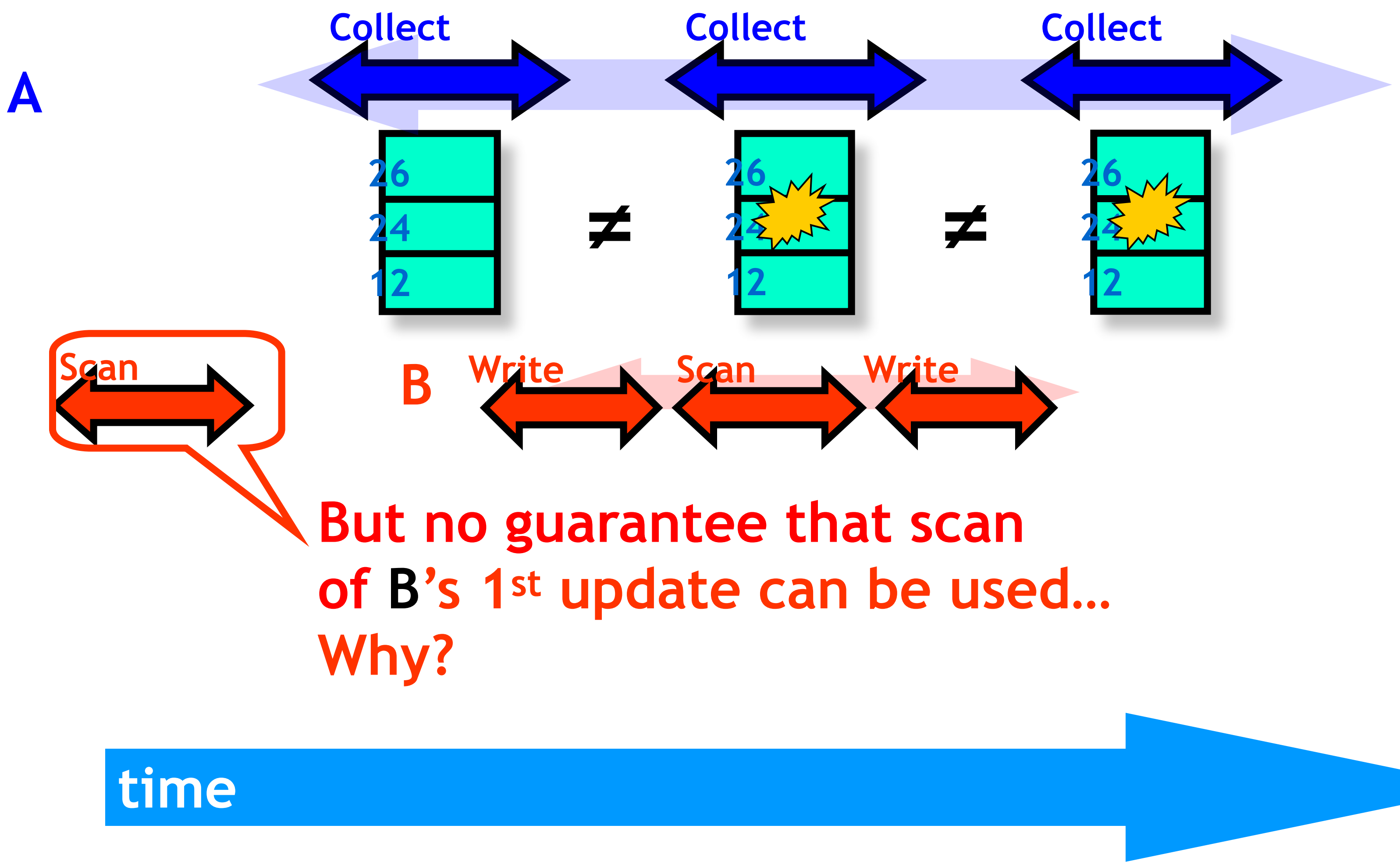
Wait-free Snapshot

B’s 1st update must have written during 1st collect



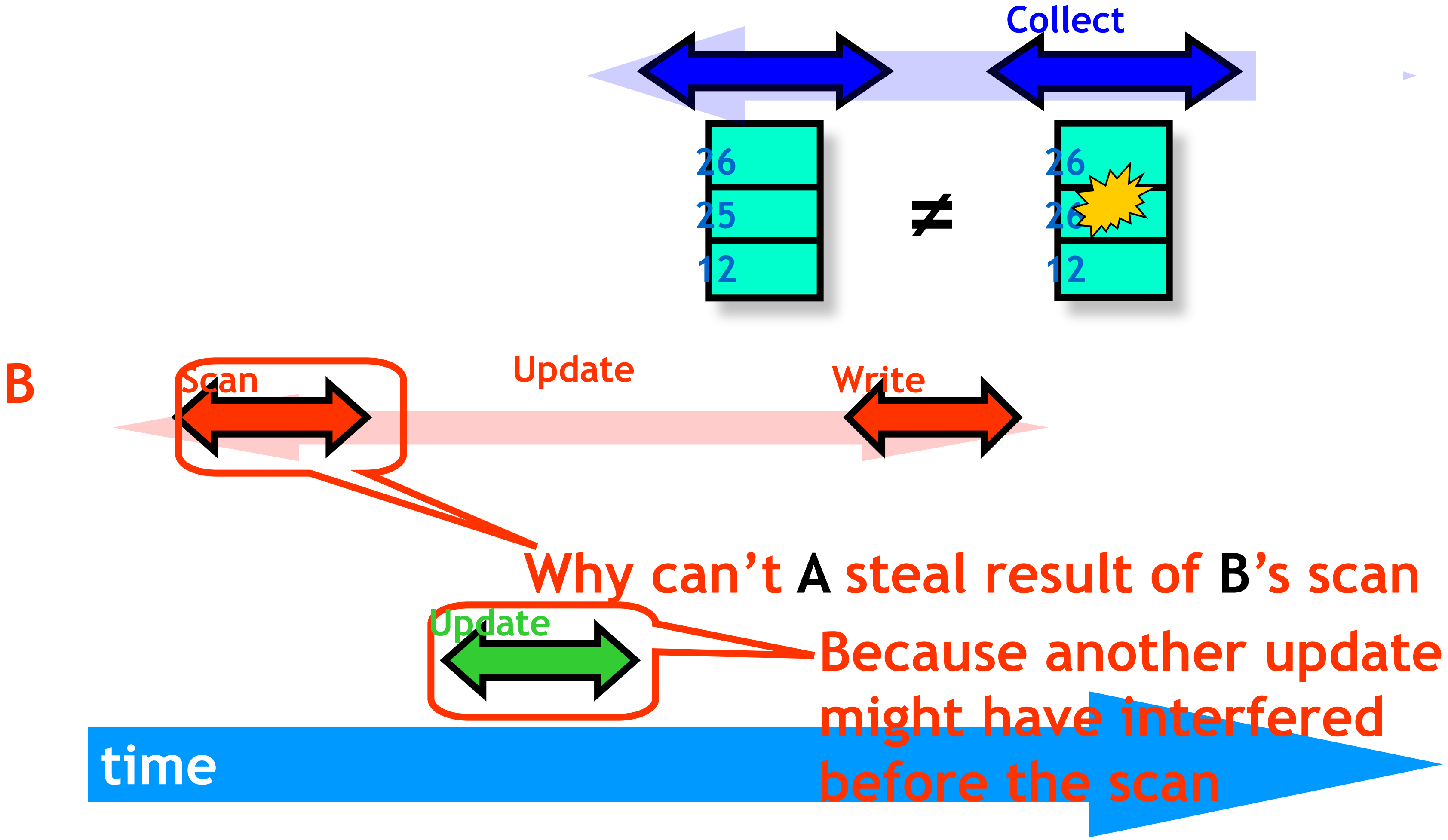
So A can steal result of B’s scan

Wait-free Snapshot



But no guarantee that scan of B’s 1st update can be used… Why?

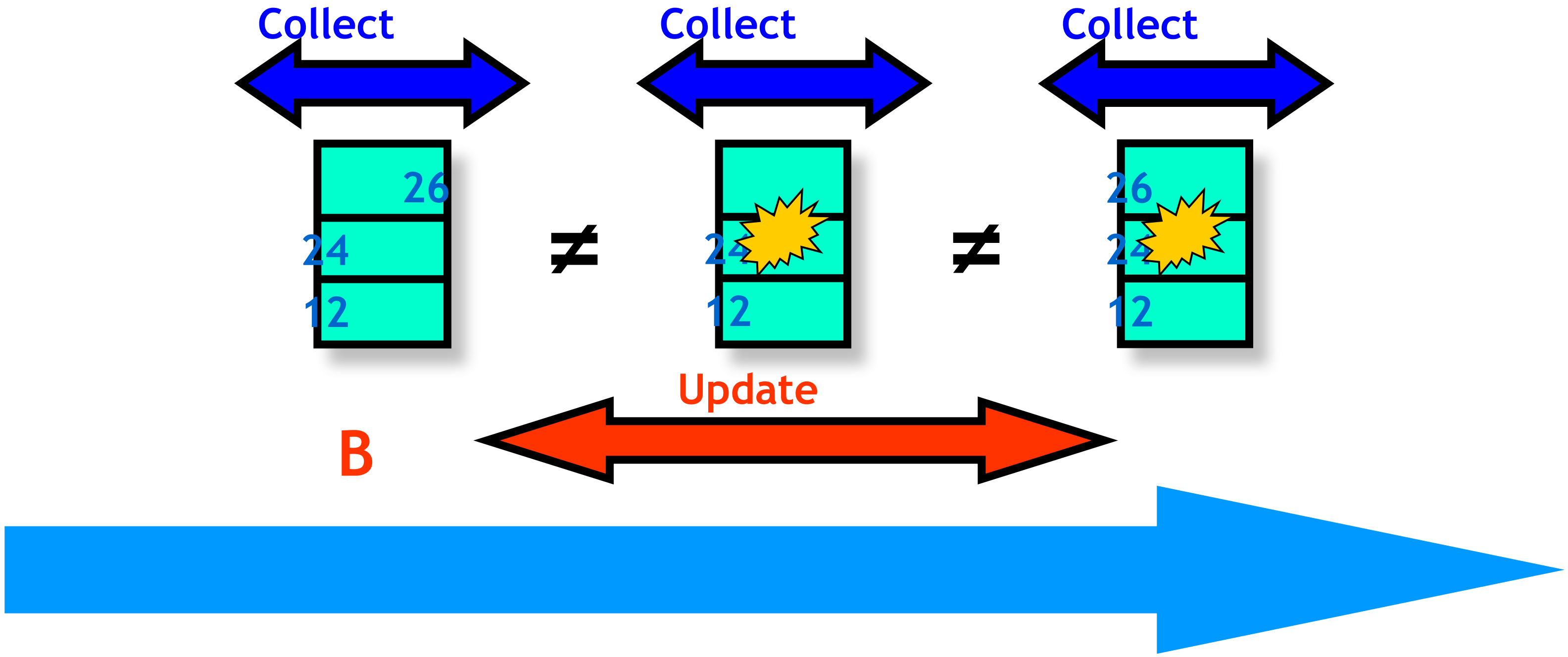
Once is not Enough



Why can't **A** steal result of **B**'s scan

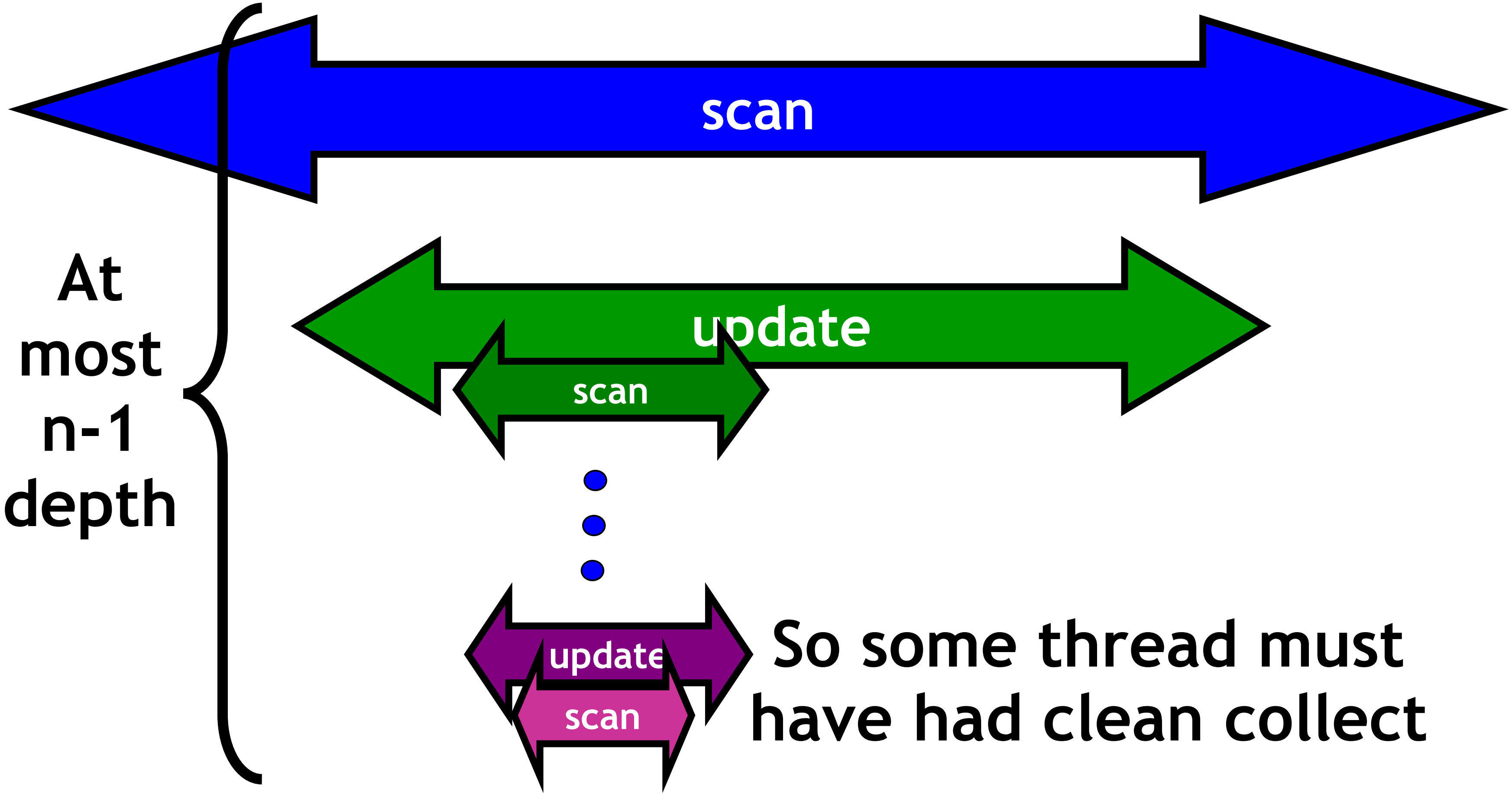
Because another update might have interfered before the scan

Someone Must Move Twice



If we collect n times…some thread
Must move twice (Pigeon hole)

Scan is Wait-free



So some thread must have had clean collect

Wait-Free Snapshot Label

```
public class SnapValue {
 public int   label;
 public int   value;
 public int[] snap;
}
```

Wait-Free Snapshot Label

```
public class SnapValue {
 public int    label;
 public int    value;
 public int[] snap;
}
```

Counter incremented with each snapshot

Wait-Free Snapshot Label

```
public class SnapValue {
 public int   label;
 public int   value;
 public int[] snap;
}
```

Actual value

Wait-Free Snapshot Label

```
public class SnapValue {
 public int    label;
 public int    value;
 public int[] snap;
}
```

most recent snapshot

Wait-free Update

```
public void update(int value) {
 int i = Thread.myIndex();
 int[] snap = this.scan();
 SnapValue oldValue = r[i].read();
 SnapValue newValue =
  new SnapValue(oldValue.label+1,
                value, snap);
 r[i].write(newValue);
 }
```

Wait-free Scan

```
public void update(int value) {
 int i = Thread.myIndex();
 int[] snap = this.scan();
 SnapValue oldValue = r[i].read();
 SnapValue newValue =
  new SnapValue(oldValue.label+1,
                value, snap);
 r[i].write(newValue);
 }
```

Take scan

Wait-free Scan

```
public void update(int value) {
 int i = Thread.myIndex();
 int[] snap = this.scan();
 SnapValue oldValue = r[i].read();
 SnapValue newValue =
  new SnapValue(oldValue.label+1,
                value, snap);
 r[i].write(newValue);
 }
```

Take scan

Label value with scan

Wait-free Scan

```
 public int[] scan() {
  SnapValue[] oldCopy, newCopy;
  boolean[] moved = new boolean[n];
  oldCopy = collect();
  collect: while (true) {
  newCopy = collect();
  for (int j = 0; j < n; j++) {
    if (oldCopy[j].label != newCopy[j].label) {
      …
  }}
  return getValues(newCopy);
}}}
```

Wait-free Scan

```
public int[] scan() {
 SnapValue[] oldCopy, newCopy;
 boolean[] moved = new boolean[n];
 oldCopy = collect();
 collect: while (true) {
 newCopy = collect();
 for (int j = 0; j < n; j++) {
   if (oldCopy[j].label != newCopy[j].label) {
     …
 }}
 return getValues(newCopy);
}}}
```

Keep track of who moved

Wait-free Scan

```
public int[] scan() {
 SnapValue[] oldCopy, newCopy;
 boolean[] moved = new boolean[n];
 oldCopy = collect();
 collect: while (true) {
 newCopy = collect();
 for (int j = 0; j < n; j++) {
   if (oldCopy[j].label != newCopy[j].label) {
     …
 }}
 return getValues(newCopy);
}}}
```

Repeated double collect

Wait-free Scan

```
public int[] scan() {
 SnapValue[] oldCopy, newCopy;
 boolean[] moved = new boolean[n];
 oldCopy = collect();
 collect: while (true) {
 newCopy = collect();
 for (int j = 0; j < n; j++) {
   if (oldCopy[j].label != newCopy[j].label) {
     …
 }}
 return getValues(newCopy);
}}}
```

If mismatch detected…lets expand here…

Mismatch Detected

```
  if (oldCopy[j].label != newCopy[j].label) {
    if (moved[j]) {       // second move
     return newCopy[j].snap;
    } else {
     moved[j] = true;
     oldCopy = newCopy;
     continue collect;
  }}}
  return getValues(newCopy);
}}}
```

Mismatch Detected

```
  if (oldCopy[j].label != newCopy[j].label) {
   if (moved[j]) {
    return newCopy[j].snap;
   } else {
    moved[j] = true;
    oldCopy = newCopy;
    continue collect;
  }}}
  return getValues(newCopy);
}}}
```

If thread moved twice, just steal its second snapshot

Mismatch Detected

```
  if (oldCopy[j].label != newCopy[j].label) {
   if (moved[j]) {      // second move
    return newCopy[j].snap;
   } else {
    moved[j] = true;
    oldCopy = newCopy;
    continue collect;
 }}}
 return getValues(newCopy);
}}}
```

Remember that thread moved

Snapshot Summary

- We saw that we can build wait-free atomic snapshot from atomic registers

Multiple Assignment Theorem

- Atomic registers cannot implement multiple assignment
- Weird or what?
  - Single location write/multi location read OK
    (= Atomic Snapshot)
  - Multi location write/single location read impossible

Proof Strategy

- If we can write to 2/3 array elements
  - We can solve 2-consensus
  - Impossible with atomic registers
- Therefore
  - Cannot implement multiple assignment with atomic registers

Proof Strategy

- Take a 3-element array
  - A writes atomically to slots 0 and 1
  - B writes atomically to slots 1 and 2
  - Any thread can scan any set of locations

Double Assignment Interface

```
interface Assign2 {
 public void assign(int i1, int v1,
                    int i2, int v2);
 public int read(int i);
}
```

Double Assignment Interface

```
interface Assign2 {
 public void assign(int i1, int v1,
                    int i2, int v2);
 public int read(int i);
}
```

Atomically assign

$value[i_1]= v_1$

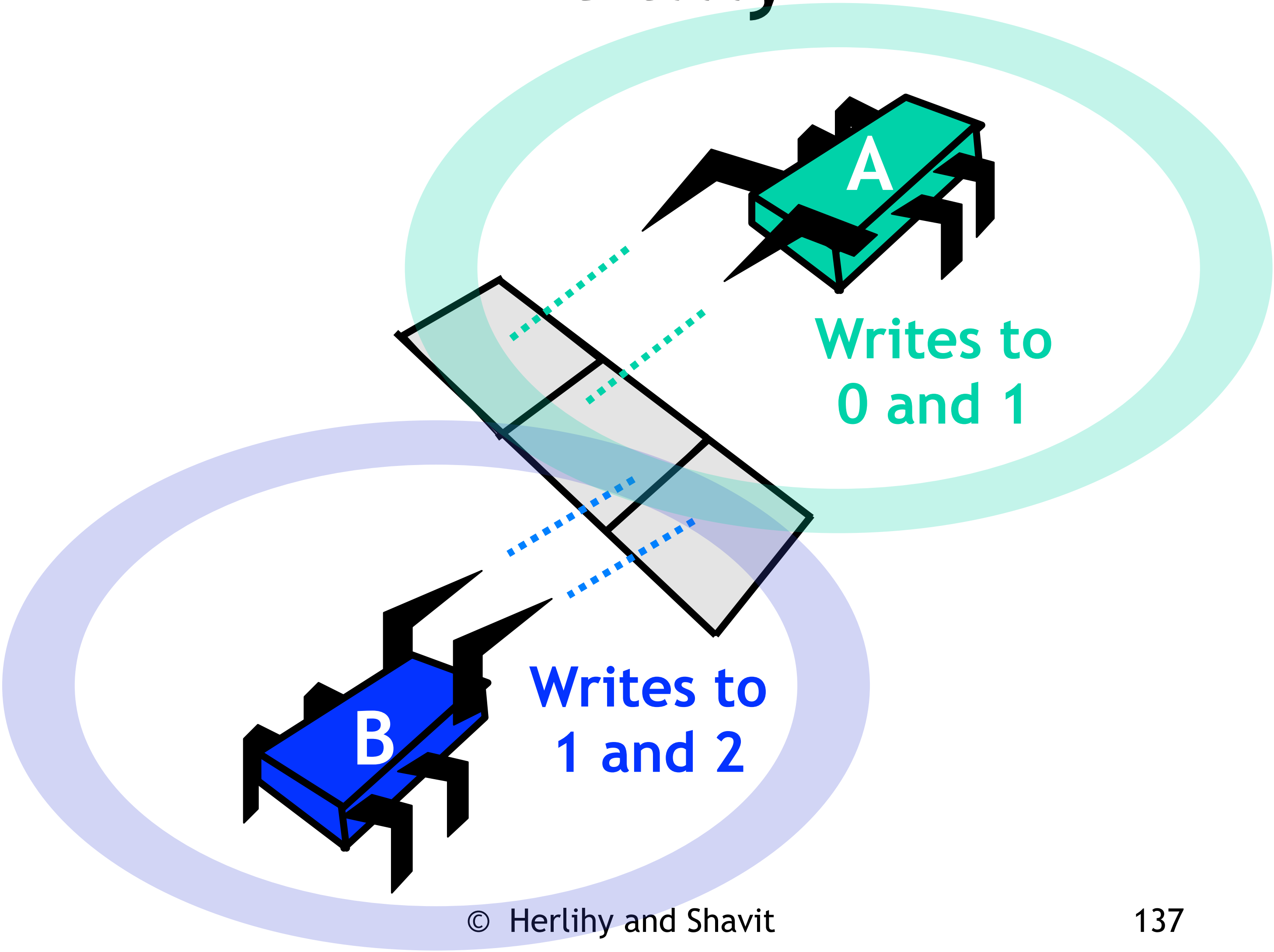
$value[i_2]= v_2$

Double Assignment Interface

```
interface Assign2 {
 public void assign(int i_1, int v_1,
                    int i_2, int v_2);
 public int read(int i);
}
```

Return i-th value

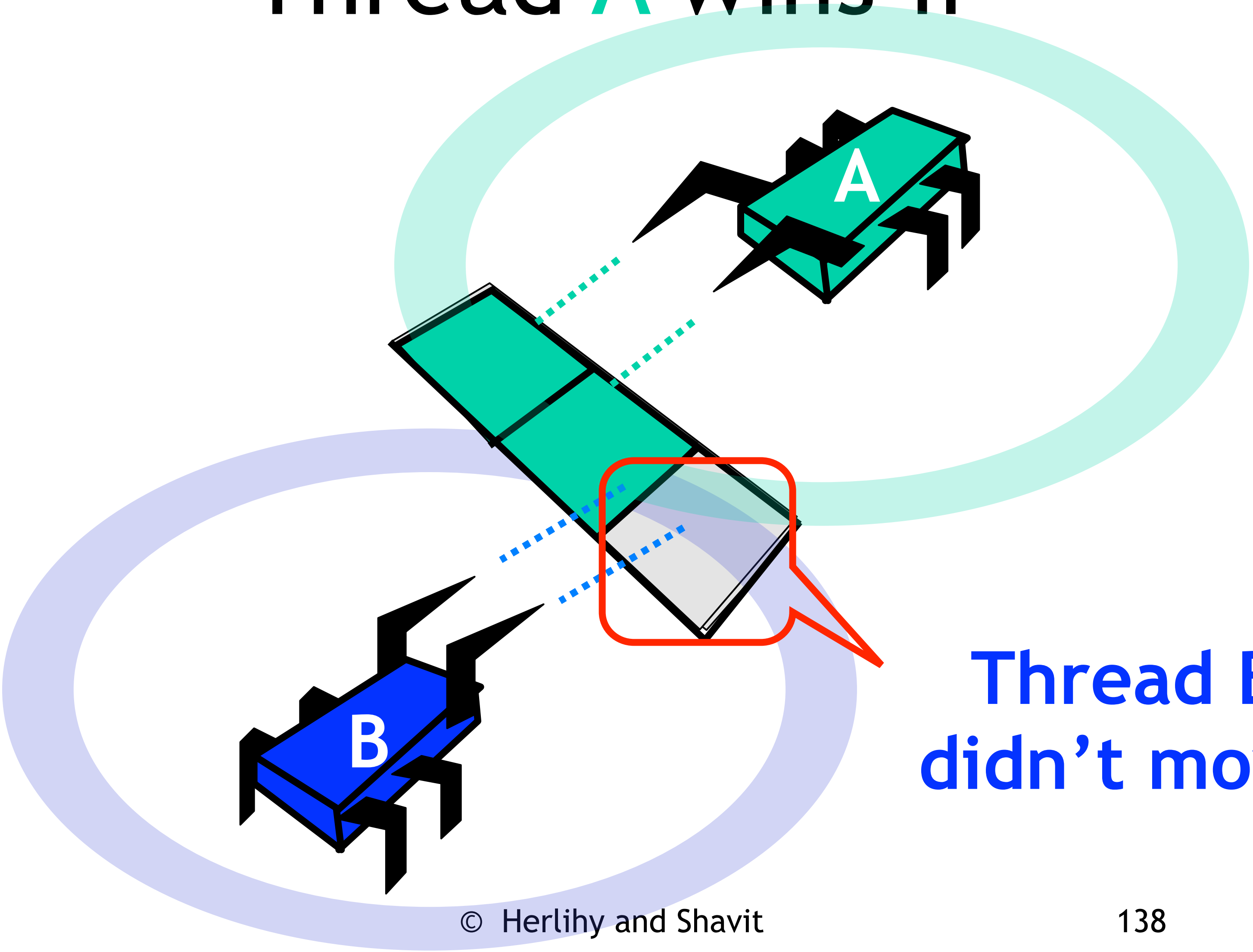
Initially



Writes to
0 and 1

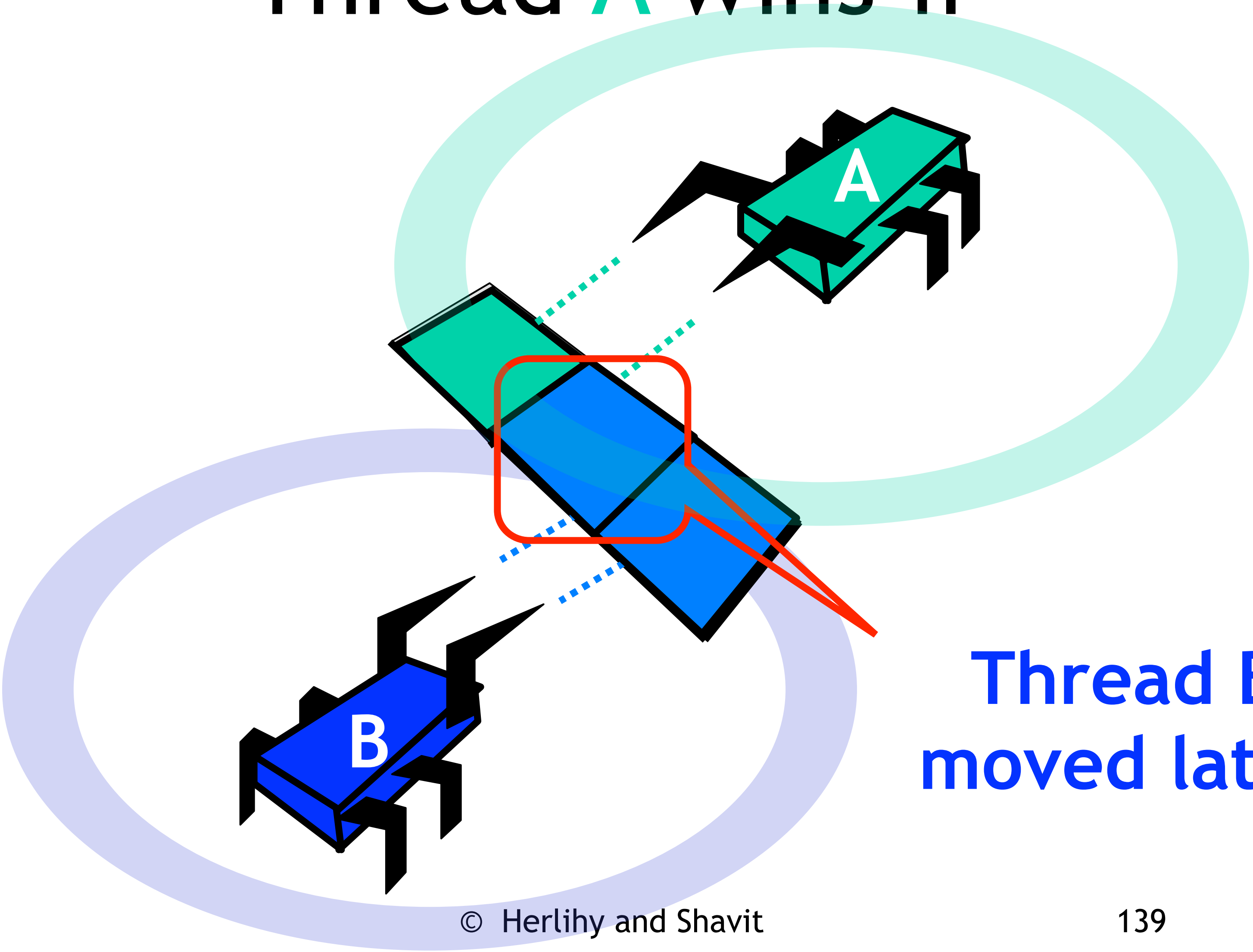
Writes to
1 and 2

Thread A wins if



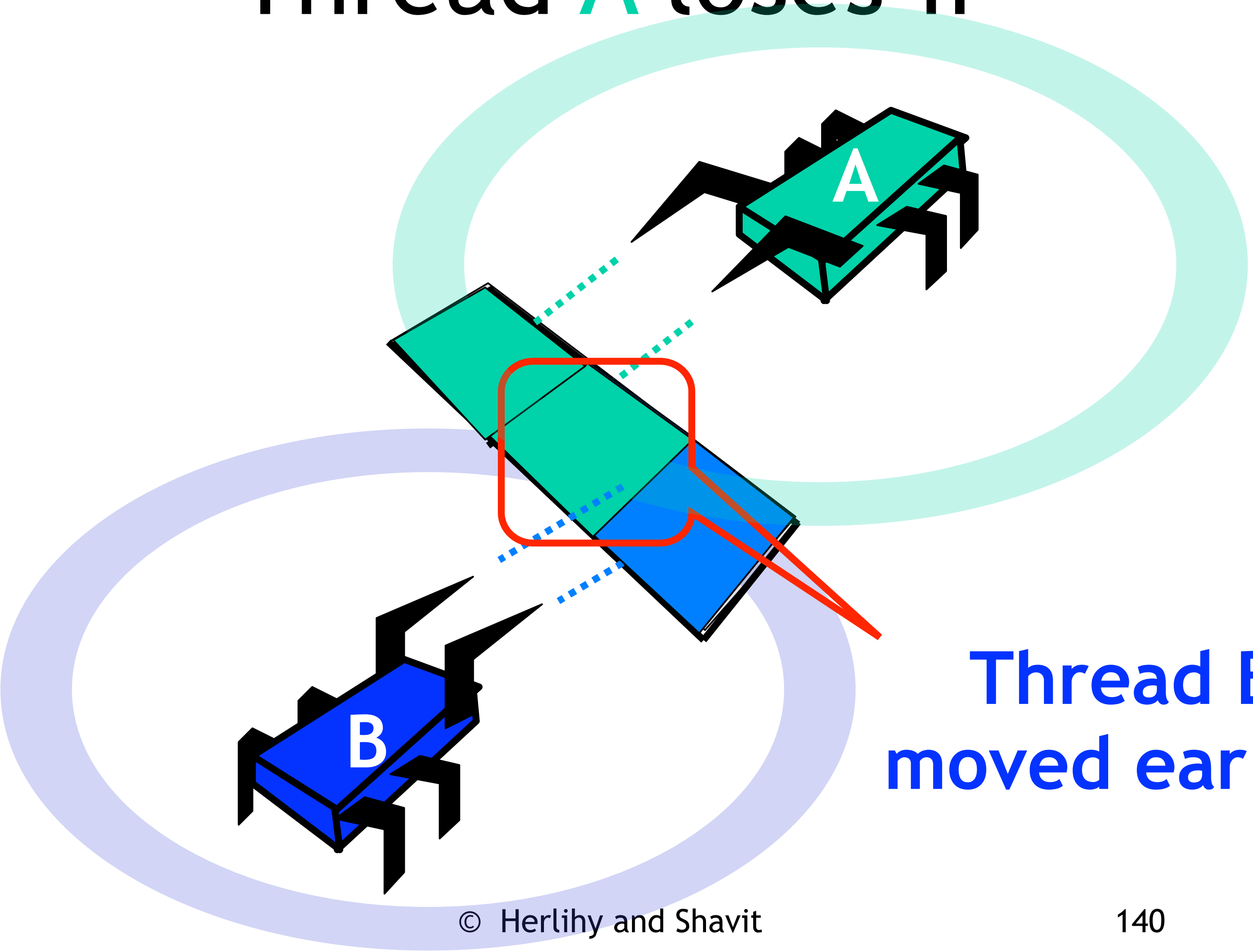
Thread B didn't move

Thread A wins if



Thread B moved later

Thread A loses if



Thread B moved earlier

Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Extends **ConsensusProtocol**

Decide sets j=1-i and proposes value

Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Three slots initialized to EMPTY

Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Assign id 0 to entries 0,1 (or id 1 to entries 1,2)

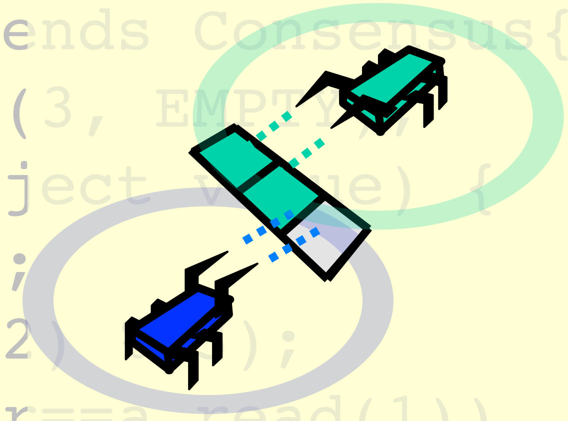
Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

Read the register my thread didn’t assign

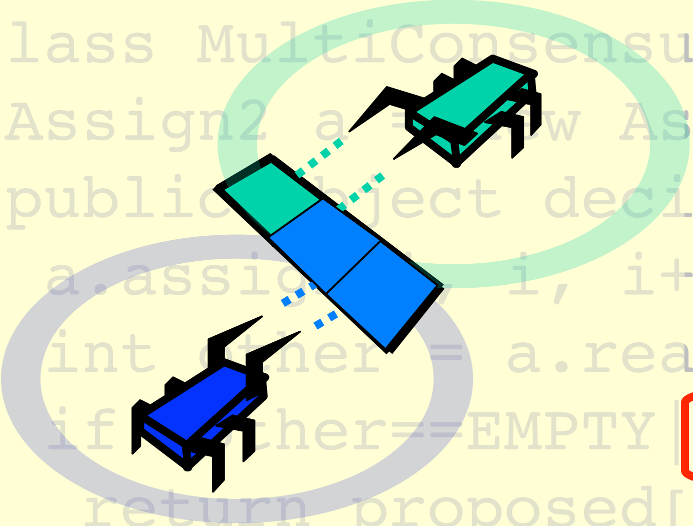
Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```



Other thread didn't move, so I win

Multi-Consensus Code



```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
}}
```

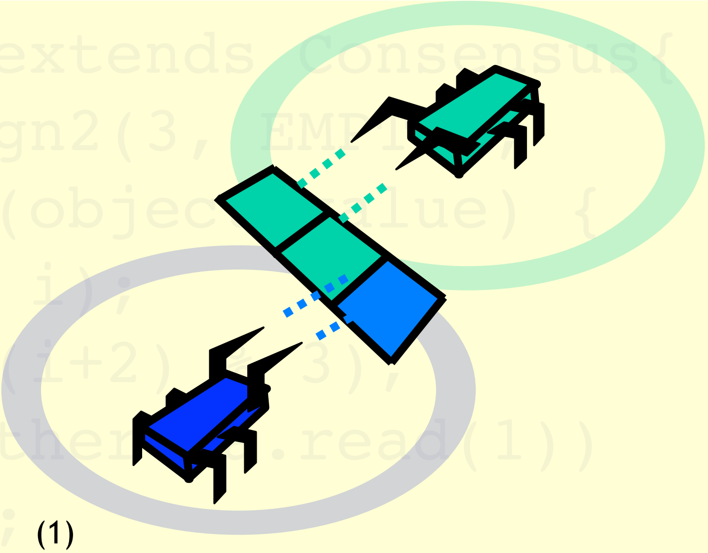
Other thread moved later so I win

Multi-Consensus Code

```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i];
  else
   return proposed[j];
  }}
```

OK, I win.

Multi-Consensus Code



```
class MultiConsensus extends Consensus{
 Assign2 a = new Assign2(3, EMPTY);
 public Object decide(object value) {
  a.assign(i, i, i+1, i);
  int other = a.read((i+2) % 3);
  if (other==EMPTY||other==a.read(1))
   return proposed[i]; (1)
  else
   return proposed[j];
 }}
```

Other thread moved first, so I lose

Summary

- If a thread can assign atomically to 2 out of 3 array locations
- Then we can solve 2-consensus
- Therefore
  - No wait-free multi-assignment from read/write registers

Read-Modify-Write Objects

- Method call
  - Returns object's prior value `x`
  - Replaces `x` with `mumble(x)`

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
  getAndMumble() {
     int prior  = this.value;
     this.value = mumble(this.value);
     return prior;
  }
}
```

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
  getAndMumble() {
   int prior  = this.value;
   this.value = mumble(this.value);
   return prior;
  }
}
```

Return prior value

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
  getAndMumble() {
    int prior  = this.value;
    this.value = mumble(this.value);
    return prior;
  }
}
```

Apply function to current value

RMW Everywhere!

- Most synchronization instructions
  - are RMW methods
- The rest
  - Can be trivially transformed into RMW methods

Example: Read

```
public abstract class RMWRegister {
  private int value;

  public int synchronized read() {
    int prior  = this.value;
    this.value = this.value;
    return prior;
  }

}
```

Example: Read

```
public abstract class RMW {
  private int value;

  public void synchronized read() {
    int prior  = this.value;
    this.value = this.value;
    return prior;
  }

}
```

Apply **f(v)=v**, the identity function

Example: getAndSet

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
   getAndSet(int v) {
  int prior  = this.value;
  this.value = v;
  return prior;
 }
 …
}
```

Example: getAndSet (swap)

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
   getAndSet(int v) {
  int prior  = this.value;
  this.value = v;
  return prior;
 }
 …
}
```

F(x)=v is constant function

getAndIncrement

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
   getAndIncrement() {
  int prior  = this.value;
  this.value = this.value + 1;
  return prior;
 }
 …
}
```

getAndIncrement

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
   getAndIncrement() {
  int prior  = this.value;
  this.value = this.value + 1;
  return prior;
 }
 …
}
```

F(x) = x+1

getAndAdd

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
   getAndAdd(int a) {
  int prior  = this.value;
  this.value = this.value + a;
  return prior;
 }
 …
}
```

Example: getAndAdd

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
   getAndIncrement(int a) {
  int prior  = this.value;
  this.value = this.value + a;
  return prior;
 }
 …
}
```

$F(x) = x+a$

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = this.value;
  if (this.value==expected) {
   this.value = update; return true;
  }
  return false;
  } … }
```

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = this.value;
  if (this.value==expected) {
   this.value = update; return true;
  }
  return false;
 } … }
```

If value is what was expected, …

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
 int prior = this.value;
 if (this.value==expected) {
  this.value = update; return true;
  }
 return false;
 } … }
```

… replace it

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = this.value;
  if (this.value==expected) {
   this.value = update;
   }
  return false;
  } … }
```

```
return true;
```

Report success

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = this.value;
  if (this.value==expected) {
   this.value = update; return true;
   }
  return false;
  } … }
```

Otherwise report failure

Read-Modify-Write

```
public abstract class RMWRegister {
 private int value;

 public void synchronized
  getAndMumble() {
    int prior  = this.value;
    this.value = mumble(this.value);
    return prior;
  }
}
```

Let’s characterize F(x)…

Definition

- A RMW method
  - With function mumble(x)
  - is non-trivial if there exists a value v
  - Such that v ≠ mumble(v)

Par Example

- `Identity(x) = x`
  - is trivial
- `getAndIncrement(x) = x+1`
  - is non-trivial

Theorem

- Any non-trivial RMW object has consensus number at least 2
- No wait-free implementation of RMW registers from atomic registers
- Hardware RMW instructions not just a convenience

Reminder

- Subclasses of `consensus` have
  - `propose(x)` method
    - which just stores `x` into `proposed[i]`
    - Built-in method
  - `decide(object value)` method
    - which determines winning value
    - Customized, class-specific method

Proof

```
public class RMWConsensus
     implements ConsensusProtocol {
 private RMWRegister r = v;
 public Object decide(object value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Proof

```
public class RMWConsensus
     implements ConsensusProtocol {
 private RMWRegister r = v;
 public Object decide(object value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Initialized to v

Proof

```
public class RMWConsensus
     implements ConsensusProtocol {
 private RMWRegister r = v;
 public Object decide(object value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Am I first?

Proof

```
public class RMWConsensus
     implements ConsensusProtocol {
 private RMWRegister r = v;
 public Object decide(object value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

Yes, return my input

Proof

```
public class RMWConsensus
     implements ConsensusProtocol {
 private RMWRegister r = v;
 public Object decide(object value) {
  propose(value);
  if (r.getAndMumble() == v)
   return proposed[i];
  else
   return proposed[j];
}}
```

No, return other's input

Proof

- We have displayed
  - A two-thread consensus protocol
  - Using any non-trivial RMW object

Interfering RMW

- Let F be a set of functions such that for all f_i and f_j, either
  - Commute: $f_i(f_j(v))=f_j(f_i(v))$
  - Overwrite: $f_i(f_j(v))=f_i(v)$
- Claim: Any set of RMW objects that commutes or overwrites has consensus number exactly 2

Examples

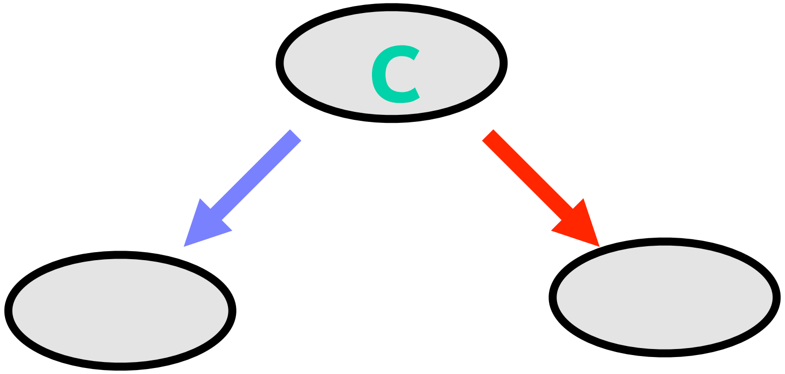
- “test-and-set” getAndSet(1) $f(v)=1$

  Overwrite $f_i(f_j(v))=f_i(v)$
- “swap” getAndSet(x) $f(v)=x$

  Overwrite $f_i(f_j(v))=f_i(v)$
- “fetch-and-inc” getAndIncrement() $f(v)=v+1$

  Commute $f_i(f_j(v))= f_j(f_i(v))$

Meanwhile Back at the Critical State



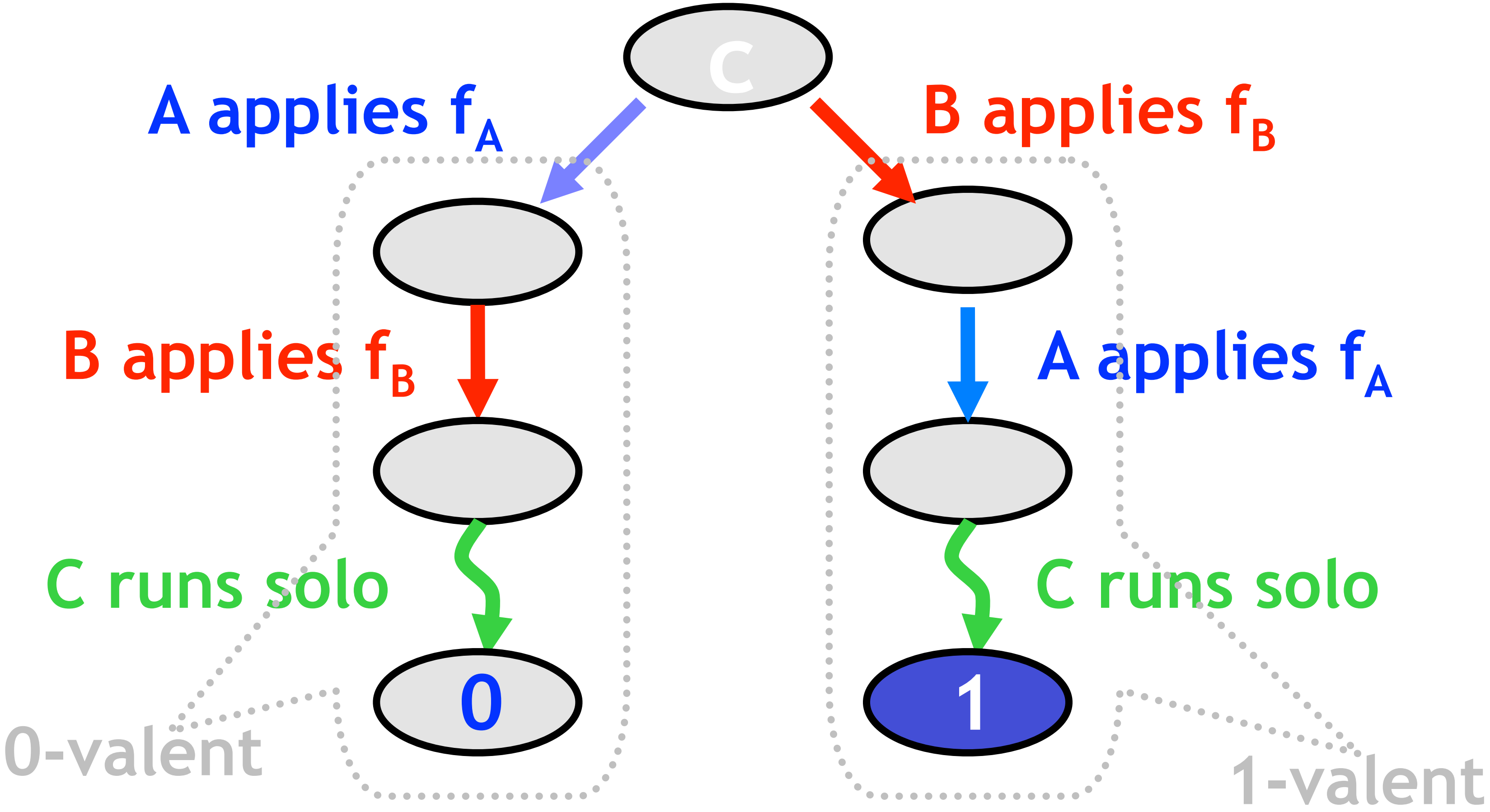
A about to apply f_A

B about to apply f_B

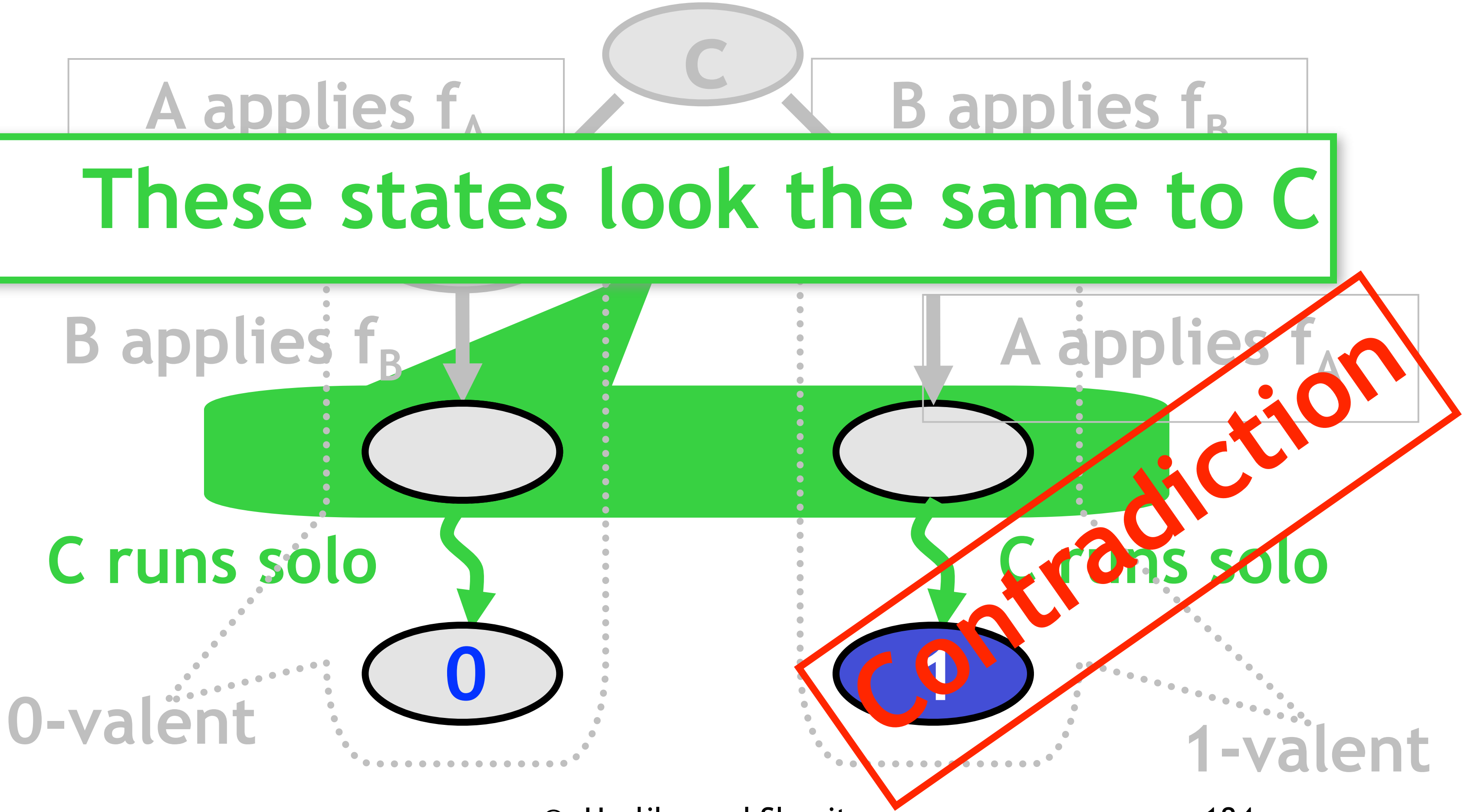
0-valent

1-valent

Maybe the Functions Commute

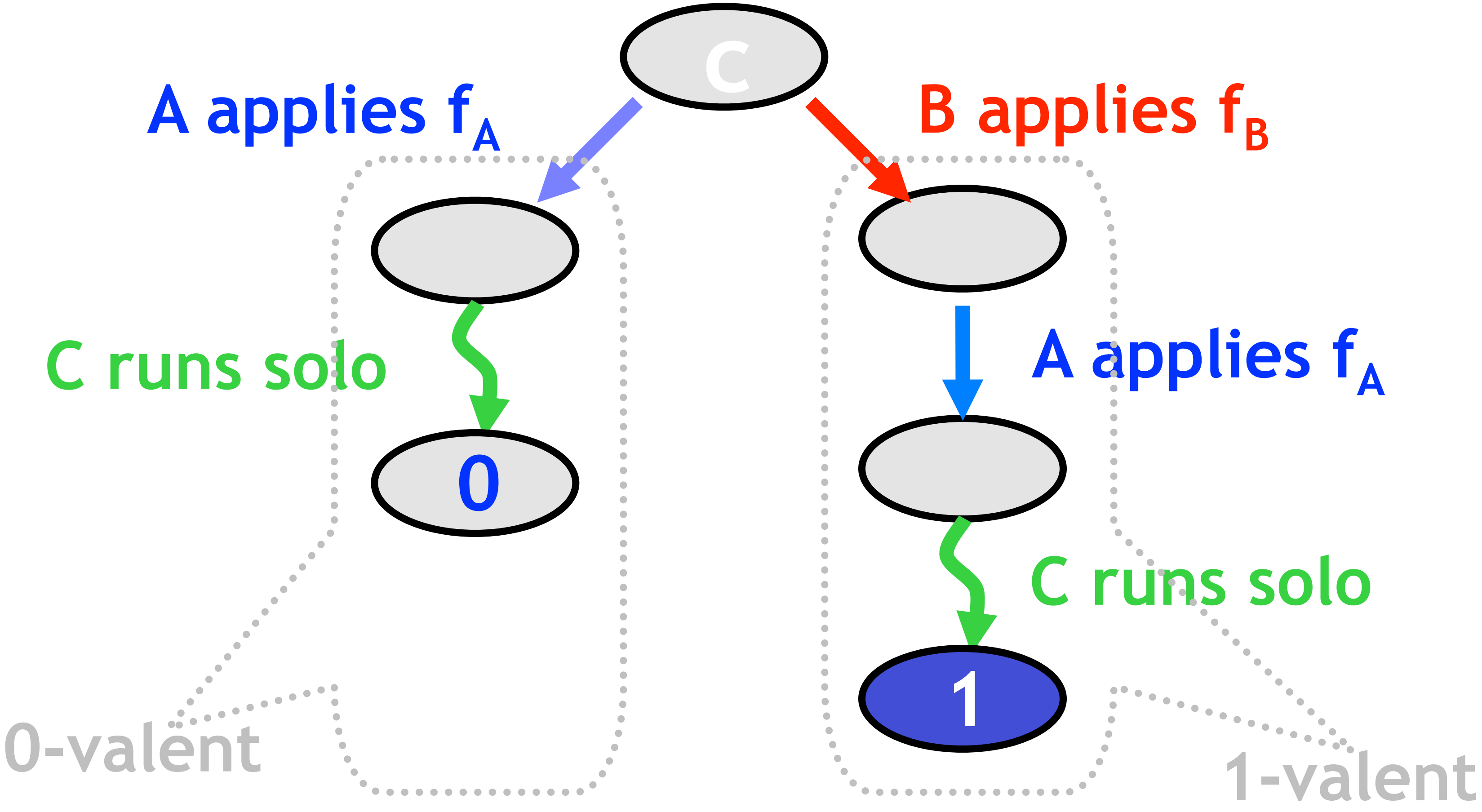


Maybe the Functions Commute

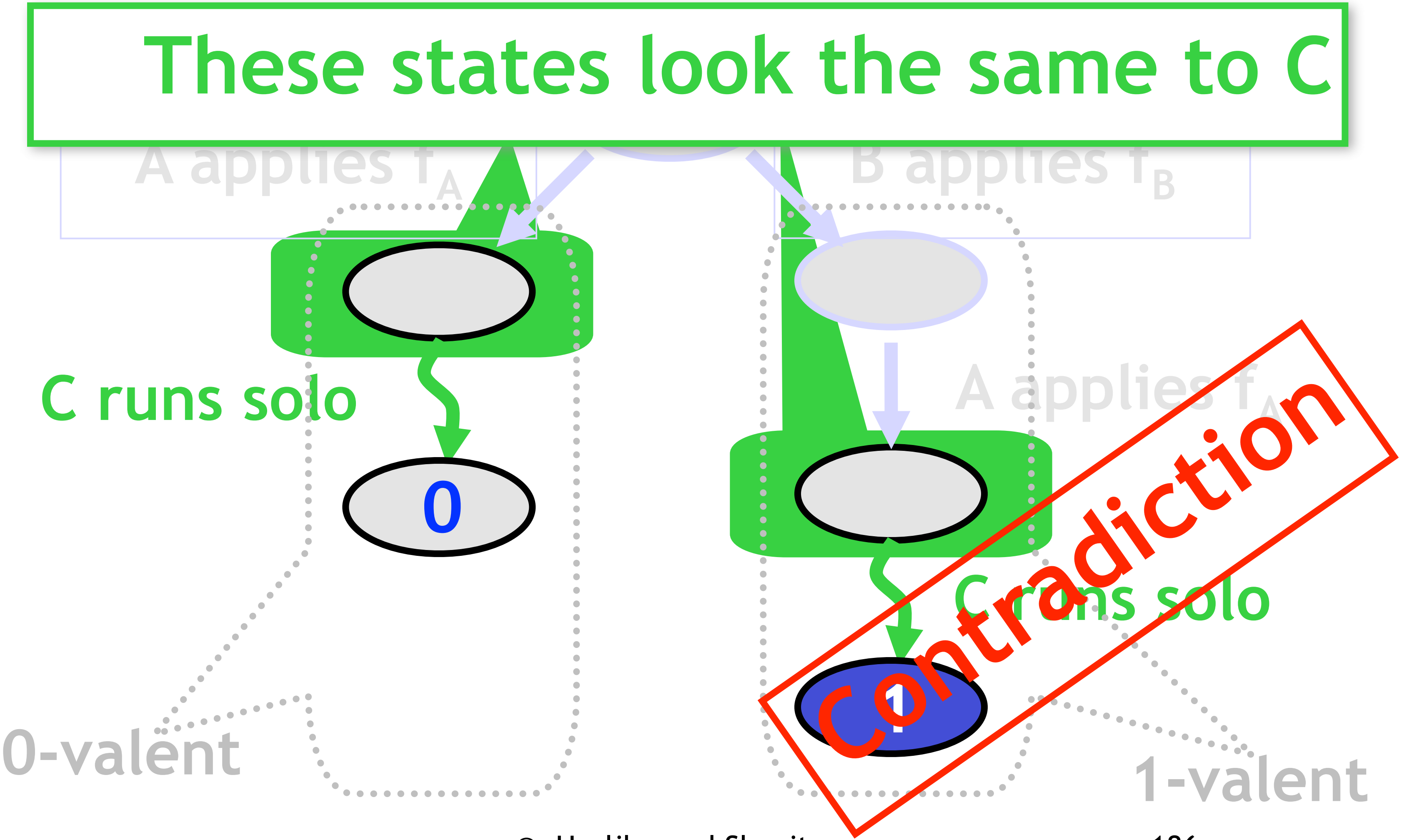


These states look the same to C

Maybe the Functions Overwrite



Maybe the Functions Overwrite



Impact

- Many early machines provided these "weak" RMW instructions
  - Test-and-set (IBM 360)
  - Fetch-and-add (NYU Ultracomputer)
  - Swap (Original SPARCs)
- We now understand their limitations
  - But why do we want consensus anyway?

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
  int prior = this.value;
  if (this.value==expected) {
   this.value = update; return true;
  }
  return false;
  } … }
```

compareAndSet

```
public abstract class RMWRegister {
 private int value;
 public boolean synchronized
   compareAndSet(int expected,
                 int update) {
 int prior = this.value;
 if (this.value==expected) {
  this.value = update; return true;
  }
 return false;
 } … }
```

replace value if its what we expected, …

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
     implements ConsensusProtocol {
 private AtomicInteger r =
   new AtomicInteger(-1);
 public Object decide(object value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
     implements ConsensusProtocol {
 private AtomicInteger r =
   new AtomicInteger(-1);
 public Object decide(object value) {
  propose(value)
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

Initialized to -1

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
     implements ConsensusProtocol {
 private AtomicInteger r =
   new AtomicInteger(-1);
 public Object decide(object value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

Try to swap in my id

compareAndSet Has ∞ Consensus Number

```
public class RMWConsensus
     implements ConsensusProtocol {
 private AtomicInteger r =
   new AtomicInteger(-1);
 public Object decide(object value) {
  propose(value);
  r.compareAndSet(-1,i);
  return proposed[r.get()];
 }
}
```

Decide winner's preference

The Consensus Hierarchy

1 Read/Write Registers, Snapshots…

2 getAndSet, getAndIncrement, …

.
.
.

∞ compareAndSet,…

Multiple Assignment

- Atomic k-assignment
- Solves consensus for 2k-2 threads
- Every even consensus number has an object (can be extended to odd numbers)

Lock-Freedom

- Lock-free: in an infinite execution infinitely often some method call finishes (obviously, in a finite number of steps)
- Pragmatic approach
- Implies no mutual exclusion

Lock-Free vs. Wait-free

- Wait-Free: each method call takes a finite number of steps to finish
- Lock-free: in an infinite execution infinitely often some method call finishes

Lock-Freedom

- Any wait-free implementation is lock-free.
- Lock-free is the same as wait-free if the execution is finite.
- Old saying: “Lock-free is to wait-free as deadlock-free is to lockout-free.”

Lock-Free Implementations

- Lock-free consensus is just as impossible
- Lock-free = Wait-free for finite executions
- **All the results we presented hold for lock-free algorithms also.**

There is More: Universality

- Consensus is universal
- From n-thread consensus we can build
  - Wait-free/Lock-free,
  - Linearizable,
  - n-threaded,
  - Implementation
  - Of any sequentially specified object